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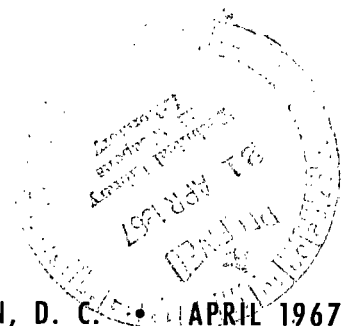


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A METHOD OF ASYMPTOTIC EXPANSIONS
FOR SINGULAR PERTURBATION PROBLEMS
WITH APPLICATION IN VISCOUS FLOW

by E. Dale Martin
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Moffett Field, Calif.



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A METHOD OF ASYMPTOTIC EXPANSIONS FOR SINGULAR
PERTURBATION PROBLEMS WITH APPLICATION

IN VISCOUS FLOW

By E. Dale Martin

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SUMMARY

A method of asymptotic expansions for solving certain classes of singular perturbation problems is presented. The method is a new approach to constructing inner and outer expansions. A classification of singular perturbation problems is given to clarify when this method is applicable and to aid in determination of an appropriate matching rule to use. Simple examples are given to illustrate use of the method.

For an important class of boundary-layer-type problems, this approach employs a stronger form of the matching principle than used previously. All displacement effects are retained explicitly in this matching process, and higher order displacements are identified and calculated. The concepts of complex displacement and multiple displacements are introduced. The development of the method leads to clarification of conditions for algebraic or exponential approach of inner to outer expansion solutions.

The expansion method developed here is applied to a model of a familiar typical physical problem in viscous-flow theory: hypersonic flow over a sphere. This application is viewed as an investigation in viscous flow of the role of displacement in matching the boundary-layer asymptotic expansion to the outer expansion. Higher order boundary-layer displacements are calculated explicitly. Results for skin friction, shock standoff distance, etc., are computed and compared with results from a numerical solution of the same model problem to demonstrate the validity of the expansion method.

INTRODUCTION

The main purpose of this paper is to outline a method for solving certain classes of singular perturbation problems including, in particular, an important class of boundary-layer-type problems. The proposed method is a new approach to defining and constructing inner and outer expansions. This approach makes possible the use of a stronger form of the matching principle than that used previously. It may be especially useful in problems where certain displacement effects are important.

Singular perturbation problems occur in virtually every branch of mathematical physics. A singular perturbation problem is encountered in seeking a mathematical solution for a limiting value (e.g., zero) of a parameter,

or in seeking a small perturbation solution, and when the solution so obtained is not uniformly valid throughout the domain of the independent variables. (For a comprehensive discussion and treatment, see ref. 1.) For example, the formal solution may become infinite at some locus of the independent (e.g., space) variables. This occurs near a rounded leading edge in the small perturbation solution of thin-airfoil theory (see ref. 2). In other problems the nonuniformity may be evidenced by a discontinuity of the solution within the region of interest (e.g., the discontinuous pressure across a shock wave in inviscid-flow theory) or by failure of the solution to satisfy a boundary condition (i.e., by a discontinuity at the boundary). The prime example of this type is in the calculation of fluid flow at high Reynolds number over a surface, in which the flow is essentially inviscid except for a thin boundary layer at the surface. The inviscid-flow solution fails to satisfy the no-slip condition at the boundary surface. The boundary-layer theory of Prandtl (ref. 3) was developed to cope with the mathematical nonuniformity in the flow calculation. Many other singular perturbation problems of this type occur in all branches of fluid mechanics (e.g., see ref. 1). The discontinuity or failure to satisfy a boundary condition is usually, but not always, due to the order of the differential equation, or set of equations, being reduced in the process of letting the small parameter become zero. The discontinuity is actually an asymptotic representation of a quick transition (rapid variation) of one or more dependent variables over a small range of the independent variable (ref. 4). In the twenty-eighth Josiah Willard Gibbs lecture in 1954, Professor Friedrichs discussed the asymptotic phenomena associated with "quick transition regions" in many branches of mathematical physics. Still another type of singular perturbation problem has a prime example in the Stokes' paradox of viscous flow, in which the approximate solution valid near a body for very small Reynolds number cannot approach the uniform-stream condition at large distances.

Rather than seek a simple limit (asymptotic solution) or simple perturbation solution for the function of interest, one may attempt to construct a perturbation expansion (power series) of the solution in terms of finite values of the small parameter. Such an expansion would yield higher approximations to the solution (if uniformly valid), which are needed for sufficient accuracy if the perturbation parameter is not very small. The small perturbation theory originated by Sir Isaac Newton has been highly developed by many others (see discussion by W. R. Sears, ref. 5). Extension of this theory to the asymptotic expansion (power series expansion in the small parameter) was devised by Poincaré (see ref. 6). If any of the terms of the expansion are not uniformly valid, ways must be found to determine an asymptotic limit function that is uniformly valid and to construct a uniformly valid asymptotic expansion extending the solution to higher order approximations for sufficient accuracy when the parameter is not very small.

A number of methods have been developed and used in finding uniformly-valid asymptotic expansions (refs. 1 and 7 describe various methods). Each method has certain attributes and certain types of problems wherein it is most advantageously used. The method of strained coordinates (also known as Lighthill's technique and as the PLK method; see ref. 8) was developed and described by Lighthill (refs. 9 and 10). This method is most useful in treating problems of the first type described above (having an infinity in the formal perturbation solution) and in singular characteristic problems (see

ref. 11). In 1953, Kuo (ref. 12) provided an extension of Lighthill's technique (see discussion by Tsien, ref. 8) to a problem in first-order boundary-layer theory: viscous flow over a finite flat plate. Lighthill's technique was used to deal with the nonuniformity at the leading edge, whereas the nonuniformity associated with the entire boundary layer (caused by the essential singularity in the complex plane of the reciprocal of Reynolds number) was dealt with by a simple stretching of variables, following Prandtl. (Kuo's treatment of that problem has been criticized by Van Dyke (refs. 1 and 13) on the grounds that it omits a concentrated force at the leading edge and that a constant in the solution was obtained by an imprecise asymptotic evaluation.)

Methods for solving singular perturbation problems of the boundary-layer type are extensions of Prandtl's boundary-layer theory, which incorporates a special kind of limiting process that gives a uniformly-valid solution near the boundary where the no-slip boundary condition is lost in the inviscid-flow theory. In 1904, Prandtl (ref. 3) recognized that the outer inviscid-flow solution could be joined to an inner solution for the nonuniform region, or boundary layer, found by properly magnifying the physical variables and by considering the relative order of magnitude of the various terms in the Navier-Stokes equations in the limit of infinite Reynolds number. Since then, "boundary-layer methods" have been applied to many other problems (see refs. 4, 8, 14, and 15; and the literature referred to in those papers). In 1935, Prandtl (ref. 16) suggested how to improve the flat-plate boundary-layer solution of Blasius (ref. 17) by iteration. This suggestion led to the use of an asymptotic expansion by Alden (ref. 18) with the first-order boundary-layer solution of Blasius as the first term. The history of that problem and correction and further extension of the solution have been treated in detail by Goldstein (ref. 19, chap. 8); the solution was further developed and discussed by Murray (ref. 20). In all boundary-layer-type problems, the general ideas of Prandtl's boundary-layer theory provide the first term in an asymptotic expansion of the solution to the full governing equations.

A significant advancement in the use of asymptotic expansions in singular perturbation problems was made by the introduction and development of a technique by Kaplun, Lagerstrom, and Cole (refs. 21-25). This technique, usually referred to as the method of "inner and outer expansions" (but more recently as "matched asymptotic expansions"), employs two distinct asymptotic series expansions of the solution: an outer expansion valid away from the nonuniformity and an inner expansion valid in the "inner region" of the nonuniformity. Limiting processes are defined according to which the inner and outer expansions must match in their "overlap region of common validity." The development of this procedure was closely related to some previous work by Latta (ref. 26). The method has been further developed and procedures established (especially for boundary-layer-type problems) by Van Dyke (refs. 13 and 27 to 31).

A method based on the ideas of inner and outer expansions is developed in the present paper. This method, which employs "displacement variables," a stronger form of the matching principle, and some new rules for obtaining appropriate forms for the expansions, may be advantageous in some problems

involving special displacement effects. The particular definitions of inner and outer functions used here and the use of displacement variables represent a different point of view from the conventional inner and outer expansion method. A result of this different point of view is that a more precise matching principle is found to apply in certain problems, which is useful in determining and understanding displacement effects. Development of the method includes introduction of the concepts of complex displacement and multiple displacements. Development of the new matching rule also leads to knowledge of conditions for which the inner expansion can approach a form of the outer expansion term by term exponentially and of the reasons for either exponential or algebraic termwise approach of inner solutions to outer solutions. The method is formulated in terms of power-series expansions of some form of the small parameter, but terms of higher order than those considered in a calculation may be of some order other than a power of the small parameter.

After the method is outlined and discussed in detail, including several simple examples to illustrate procedures, it is applied to a simplified model of the familiar problem of hypersonic viscous flow over a sphere at high Reynolds number. That problem has been studied by a number of investigators, but the main purposes in this application will be to illustrate the precise determination of displacement effects, to identify and calculate the higher order displacements themselves, and possibly to generate an increased understanding, by the point of view of this method, of the role played by displacement in the relationship between inner and outer solutions.

CHARACTERISTICS OF SEVERAL CLASSES OF SINGULAR PERTURBATION PROBLEMS

This section discusses several different classes of singular perturbation problems and their distinctions. The main purpose is to make clear the place of the method discussed below in singular perturbation theory. Since different types of nonuniformities have different characteristics and must be handled somewhat differently, it is convenient to classify problems according to the type of nonuniformity. This classification is necessary to the discussion of different types of inner and outer solutions, of the ways the inner solution can approach the outer solution, and, consequently, of the matching principles relating inner and outer solutions. Some important aspects of this classification, and of the examples given, are developed and discussed more fully in later sections.

Purely for purposes of reference in this study, different types of singular perturbation problems will arbitrarily be denoted by class 1, class 2, etc. There are classes other than those discussed here and many problems overlap the classes, that is, may have the characteristics of two classes at the same time.

The problems used here to illustrate different classes will involve only ordinary differential equations, and will be very simple. However, the classes of problems these examples represent, and the asymptotic phenomena they exemplify, are extremely important. The same classes include problems

with more than one independent variable and with both linear and nonlinear equations. The primary use of expansion methods is in finding analytical solutions to those more difficult problems. Understanding of the simpler problems can lead ultimately to the successful solution of the more difficult problems with at least some of the same essential features.

In the following, a dependent variable f may be a function of several independent variables, including a small parameter ϵ :

$$f = f(x, y, z, \epsilon)$$

The parameter ϵ may be thought of as a complex variable and f as a complex function of ϵ , even though ϵ and f will ultimately take on only real values in the solution. The theory of complex variables and, in particular, of analytic functions can then be used in discussing the behavior of f as ϵ varies, for example, as ϵ approaches zero (cf. ref. 1). Since the examples that follow will contain only derivatives with respect to y , they will be written as ordinary differential equations for $f(y)$ with $d(\)/dy$ denoted by $(\)'$.

The various classes, to be illustrated below, will be distinguished by properties I, II, and III, described by the following statements (where either I(a) or I(b) describes property I, etc.):

I. Order of the differential equation

- (a) is not reduced upon letting $\epsilon \rightarrow 0$.
- (b) is reduced upon letting $\epsilon \rightarrow 0$.

II. A discontinuity or loss of a boundary condition in the formal asymptotic solution obtained for $\epsilon \rightarrow 0$

- (a) does not occur, but the terms of the asymptotic expansion are singular (infinite) at some y (e.g., $y = 0$); that expansion solution is then designated as an "outer expansion."
- (b) occurs because of a singularity in that expansion solution (then designated "outer expansion") at (or near, as $\epsilon \rightarrow 0$) the location where the lost condition was to have been applied. (The singularity may not be apparent in lower order terms of the expansion.)
- (c) occurs, but the terms of the expansion solution (then designated "outer expansion," to be more completely defined later) are otherwise regular (i.e., are sectionally continuous and have sectionally continuous, or one-sided, derivatives) at the location where the condition is lost ($y = 0$).
- (d) occurs and the limit either of the function or of its derivatives does not exist as $\epsilon \rightarrow 0$, so there is no "outer solution."

III. An essential singularity (of the function f) with respect to ϵ on the plane $\epsilon = 0$ in f, y, ϵ space

- (a) does not exist; an appropriate inner solution is equivalent to f , and its asymptotic expansion for small ϵ (the inner expansion, a valid representation for f near $y = 0$; to be defined) approaches algebraically to the outer expansion term by term.
- (b) exists; an appropriate inner solution function valid near $y = 0$ (to be defined) approaches exponentially to the outer solution function as $y \rightarrow \infty$; but the asymptotic expansion of the inner solution approaches algebraically to the outer expansion term by term.
- (c) exists; and an inner solution (a valid representation for f near $y = 0$; to be defined) and its asymptotic expansion for small ϵ approach exponentially to a form of the outer solution term by term.
- (d) exists; but there is no "inner region" (boundary layer, or quick transition region) to which the nonuniformity is confined, and hence no possible matching of an inner solution with the outer solution.

For convenience, the following table lists the properties of the six classes to be discussed.

<u>Class</u>	<u>Properties</u>
1	Ia, IIa, IIIa
2	Ia, IIb, IIIa
3	Ib, IIb, IIIa
4	Ib, IIb, IIIb
5	Ib, IIc, IIIc
6	Ib, IId, IIId

Class 1 will denote the general class of problems for f in which statements Ia, IIa, and IIIa apply. To illustrate this class, consider example problem 1:

$$\left. \begin{aligned} (y + \epsilon f)f' + f &= 1 \\ f(1) &= 2 \end{aligned} \right\} \quad (1)$$

This is a special case of a problem used by Lighthill (ref. 10) in illustrating the method of strained coordinates and is identical to one used by Van Dyke (ref. 1). To study the characteristics of this problem, we note that the exact solution is

$$f = \left[\left(\frac{y}{\epsilon} \right)^2 + 2 \left(\frac{1+y}{\epsilon} \right) + 4 \right]^{1/2} - \frac{y}{\epsilon} \quad (2)$$

which may be expanded, using the binomial theorem, for small ϵ as:

$$f = \left(1 + \frac{1}{y} \right) + \epsilon \left(\frac{3}{2y} - \frac{1}{y^2} - \frac{1}{2y^3} \right) + \epsilon^2 \left(-\frac{3}{2y^2} - \frac{1}{2y^3} + \frac{3}{2y^4} + \frac{1}{2y^5} \right) + O(\epsilon^3) \quad \left. \begin{aligned} & \\ & \text{as } \epsilon \rightarrow 0 \end{aligned} \right\} \quad (3)$$

The expansion (3) could also be obtained directly by seeking a perturbation-expansion solution of equations (1), that is, by assuming the solution has the form

$$f = f_1(y) + \epsilon f_2(y) + \epsilon^2 f_3(y) + \dots \quad (4)$$

and substituting into equations (1), letting $\epsilon \rightarrow 0$ successively to obtain the respective problems for f_1, f_2, f_3 , etc. From equation (3), the solution apparently is singular as $\epsilon \rightarrow 0$ and $y \rightarrow 0$. Also from equation (3) we see that f is analytic for all other y at $\epsilon = 0$, and from equation (2) we see that f is analytic for all other ϵ at $y = 0$. Thus, the nonuniformity is limited to the neighborhood of the line $y = 0, \epsilon = 0$ in f, y, ϵ space, and the function f is, in general, analytic for $\epsilon = 0$ except on that line. Finding an expansion valid near $y = 0$ for small ϵ will be discussed later.

Let class 2 denote the class of problems for f in which statements Ia, IIb, and IIIa apply. An example illustrating this class is example problem 2:

$$\left. \begin{aligned} (\epsilon + y)f'' + (\alpha + 1)f' &= 0, & (0 < \alpha < \infty) \\ f(0) &= 0, & f(\infty) = 1 \end{aligned} \right\} \quad (5)$$

The exact solution is

$$f = 1 - \left(\frac{\epsilon}{\epsilon + y} \right)^\alpha \quad (6)$$

for which the expansion for small ϵ is

$$f = 1 - \left(\frac{\epsilon}{y} \right)^\alpha \left[1 - \alpha \left(\frac{\epsilon}{y} \right) + \frac{\alpha(\alpha+1)}{2!} \left(\frac{\epsilon}{y} \right)^2 + \dots \right] \quad (7a)$$

$$= 1 - \epsilon^\alpha \left(\frac{1}{y} \right)^\alpha + \epsilon^{\alpha+1} \left(\frac{\alpha}{y^{\alpha+1}} \right) - \epsilon^{\alpha+2} \left[\frac{\alpha(\alpha+1)}{2y^{\alpha+2}} \right] + \dots \quad (7b)$$

The asymptotic solution cannot satisfy the boundary condition at $y = 0$. Note that in this example f is not analytic with respect to ϵ at $\epsilon = 0$ for any fixed y unless α is a positive integer. However, if $\alpha \equiv m/n$ where m and n are positive integers, then f is analytic with respect to $\epsilon' \equiv \epsilon^{1/n}$ at $\epsilon = 0$. Further interesting properties are introduced by the variation of the parameter α , to be seen later.

Subclasses of class 2 have special properties that are of great interest in singular perturbation theory. Let class 2a denote the problems of class 2 for which the singularity in statement IIb is an essential singularity at a finite location in y space where the boundary condition is lost (without loss of generality, say at $y = 0$). Consider example problem 2a:

$$\left. \begin{aligned} y^2 f'' + (2y - \epsilon) f' &= 0 \\ f(0) &= 1, \quad f(\infty) = 0 \end{aligned} \right\} \quad (8)$$

for which the solution is

$$f = 1 - e^{-\epsilon/y} \quad (9)$$

This function has an essential singularity at $y = 0$, and the respective terms of its expansion for small ϵ do not satisfy the boundary condition at $y = 0$, although the complete solution does. This solution function can also be said to have a boundary layer at $y = \infty$ for large ϵ , which can be seen by using the transformation $z = 1/y$, $\epsilon = 1/\beta$, to obtain a problem and solution having the same forms for $f(z, \beta)$ as the function $f(y, \epsilon)$ of example problem 5a below.

Now further denote by class 2b the subclass for which the essential singularity is at $y = \infty$, where a boundary condition cannot be satisfied. This subclass is exemplified by "Stokes' paradox" of viscous flow, and is simply illustrated by example problem 2b:

$$\left. \begin{aligned} f'' + \epsilon f' &= 0 \\ f(0) &= 0, \quad f(\infty) = 1 \end{aligned} \right\} \quad (10)$$

with the solution

$$\begin{aligned} f &= 1 - e^{-\epsilon y} \\ &= \epsilon y - \frac{\epsilon^2 y^2}{2!} + \frac{\epsilon^3 y^3}{3!} - \dots \end{aligned} \quad (11)$$

The expansion of this solution for small ϵ has a nonuniformity at $y = \infty$; the boundary condition there cannot be satisfied by a finite number of terms of the expansion for small ϵ . The algebraic approach (in property IIIa) of the "inner solution" of the nonuniform region near $y = \infty$ (or near $z = 1/y = 0$) toward a finite number of terms of the "outer solution" (the expansion of f for small ϵ) as the outer region is approached ($1/\epsilon y \rightarrow \infty$) is best observed by transforming the problem by $z = 1/y$ to the form of example problem 2a above. The function (equation (11)) also has a boundary layer for large ϵ at $y = 0$, as is seen by substituting $\beta = 1/\epsilon$ to get the same form for small β as in example problem 5a below.

Denote by class 3 the class of problems for f in which statements Ib, IIb, and IIIa are true. An illustration of class 3 is provided by example problem 3:

$$\left. \begin{aligned} \epsilon f'' + (f')^{1+\mu} &= 0, \quad (0 < \mu < 1) \\ f(0) &= 0, \quad f(\infty) = 1 \end{aligned} \right\} \quad (12)$$

The asymptotic solution found by letting $\bar{\epsilon} \rightarrow 0$ cannot satisfy the boundary condition at $y = 0$. The exact solution is the same as that for example problem 2, given by equation (6), where:

$$\alpha \equiv \frac{1-\mu}{\mu} ; \epsilon \equiv \left(\frac{1-\mu}{\mu} \right) \left(\frac{\bar{\epsilon}}{1-\mu} \right)^{\frac{1}{1-\mu}} \quad (13)$$

Let class 4 be the class of problems for f in which properties Ib, IIb, and IIIb apply. Consider example problem 4a:

$$\left. \begin{aligned} \epsilon^2 f'' + (4y^2 - 6\epsilon)(1 - f) &= -2\epsilon \\ f(0) = f'(0) &= 0 \end{aligned} \right\} \quad (14)$$

The exact solution is

$$f = \left(\frac{2y}{\epsilon^{1/2}} \right) e^{-y^2/\epsilon} \int_0^{y/\epsilon^{1/2}} e^{t^2} dt \quad (15a)$$

This function has an essential singularity at $\epsilon = 0$, but has the asymptotic expansion for large y and small ϵ (and the outer expansion):

$$f \sim 1 + \frac{\epsilon}{2y^2} + \frac{1 \cdot 3 \epsilon^2}{(2y^2)^2} + \frac{1 \cdot 3 \cdot 5 \epsilon^3}{(2y^2)^3} + \dots \quad (15b)$$

Another interesting problem in this class is example problem 4b:

$$\left. \begin{aligned} \epsilon y^2 f'' + 2y^3 f' - 2\epsilon f &= 2y^3 - 4\epsilon y \\ f(0) = 0, \quad f'(0) &= 2 \end{aligned} \right\} \quad (16)$$

with the solution

$$f = y + \left(\frac{\epsilon}{y} \right) (1 - e^{-y^2/\epsilon}) \quad (17)$$

which is regular at $y = 0$, but has the outer solution, $y + \epsilon/y$.

Let class 5 denote the class of problems for f in which statements Ib, IIc, and IIIc apply. A large portion of the quick-transition phenomena of mathematical physics, including the classical boundary-layer theory, generally fits into this category. A number of examples will be given to illustrate various phenomena in this class.

One of the simplest problems with which to illustrate class 5 is the same as equations (12) but with $\mu = 0$. With this example one can observe the transition from a problem in class 3 with algebraic approach of the inner solution to a finite number of terms of the outer solution (with no essential singularity in the solution at $\epsilon = 0$) to one in class 5 with exponential

termwise approach of the inner solution to the outer solution and with an essential singularity in the solution at $\epsilon = 0$. Consider example problem 5a:

$$\left. \begin{aligned} \bar{\epsilon} f'' + f' &= 0 \\ f(0) &= 0, \quad f(\infty) = 1 \end{aligned} \right\} \quad (18)$$

The exact solution is simply

$$f = 1 - e^{-y/\bar{\epsilon}} \quad (19)$$

It is instructive to obtain this same result from equation (6), using equations (13) and letting $\mu \rightarrow 0$. In so doing, one needs the limit process:

$$\begin{aligned} \lim_{\mu \rightarrow 0} \left(\frac{\epsilon}{\epsilon + y} \right)^{\frac{1-\mu}{\mu}} &= \lim_{\mu \rightarrow 0} \left(1 + \frac{\mu y}{\epsilon} \right)^{-\frac{1}{\mu}} = \lim_{\mu \rightarrow 0} e^{-\frac{1}{\mu} \log \left(1 + \frac{\mu y}{\epsilon} \right)} \\ &= \exp \left[-\lim_{\mu \rightarrow 0} \left(\frac{\log(1 + \mu y/\epsilon)}{\mu} \right) \right] = e^{-\frac{y}{\epsilon}} \end{aligned} \quad (20)$$

Other simple problems with properties appropriate to class 5, each with certain features that will be of interest and useful to study, are the following:

Example problem 5b will be used to illustrate the common occurrence of simple first-order displacement:

$$\left. \begin{aligned} \epsilon f''' + y f'' &= 0 \\ f(0) = f'(0) &= 0; \quad f'(\infty) \sim 1 \end{aligned} \right\} \quad (21)$$

The solution is

$$f = y \operatorname{erf} \left[\frac{y}{(2\epsilon)^{1/2}} \right] + \left(\frac{2\epsilon}{\pi} \right)^{1/2} \left(e^{-y^2/2\epsilon} - 1 \right) \quad (22a)$$

where

$$\operatorname{erf} \xi \equiv 2\pi^{-1/2} \int_0^\xi e^{-t^2} dt \quad (22b)$$

The displacement (to be defined) is

$$\delta = \epsilon^{1/2} \left(\frac{2}{\pi} \right)^{1/2} \quad (22c)$$

Higher order displacement effects will be illustrated by example problem 5c:

$$\left. \begin{aligned} \epsilon f''' + f'' &= 1 \\ f(0) = f'(0) &= 0 ; \quad f'(\infty) \sim 1 + y \end{aligned} \right\} \quad (23)$$

The solution is

$$f = y + \frac{y^2}{2} - \epsilon + \epsilon e^{-y/\epsilon} \quad (24a)$$

and the displacement (to be defined) is

$$\delta = \epsilon \left(1 - \frac{1}{2} \epsilon + \frac{1}{2} \epsilon^2 - \frac{5}{8} \epsilon^3 + \dots \right) \quad (24b)$$

The displacement in a given problem may be complex although both the outer and inner solutions are purely real. An example with a purely imaginary displacement (to be discussed) is example problem 5d:

$$\left. \begin{aligned} \epsilon f''' + f'' &= 1 \\ f(0) = 0 ; \quad f'(0) &= 1 ; \quad f' \sim y + o(1) \quad \text{as } y \rightarrow \infty \end{aligned} \right\} \quad (25)$$

with the solution

$$f = \frac{y^2}{2} + \epsilon - \epsilon e^{-y/\epsilon} \quad (26)$$

Example problem 5e contains two different displacements (one of which is complex):

$$\left. \begin{aligned} \epsilon f'''' + f''' &= 0 \\ f(0) = f'(0) &= f''(0) = 0 \\ f' \sim y + o(y) &\quad \text{as } y \rightarrow \infty \end{aligned} \right\} \quad (27)$$

The solution is

$$f = \frac{y^2}{2} - \epsilon y + \epsilon^2 - \epsilon^2 e^{-y/\epsilon} \quad (28)$$

The cause of more than one displacement and the handling of such problems will be discussed.

A problem in which the asymptotic inner solution is not equivalent to the exact solution (to be discussed when the inner solution is defined) is example problem 5f:

$$\left. \begin{aligned} \epsilon f'' - f &= -1 \quad (0 \leq y \leq 1) \\ f(0) &= 2 , \quad f'(1) = 0 \end{aligned} \right\} \quad (29)$$

This mathematical model, which corresponds to a problem in one-dimensional heat conduction, was posed by Professor W. C. Reynolds of Stanford University (private communication). A similar problem was discussed by Friedrichs and by Van Dyke (see ref. 1, p. 79). The exact solution to equation (29) is

$$f = 1 + \frac{\cosh(y-1)/\epsilon^{1/2}}{\cosh 1/\epsilon^{1/2}} = 1 + \frac{e^{-y/\epsilon^{1/2}}}{1 + e^{-2/\epsilon^{1/2}}} + \frac{e^{(y-2)/\epsilon^{1/2}}}{1 + e^{-2/\epsilon^{1/2}}} \quad (30a)$$

The asymptotic (inner) solution is

$$f \sim 1 + e^{-y/\epsilon^{1/2}} + \exp \quad \text{as } \epsilon \rightarrow 0 \text{ in } 0 \leq y \leq 1 \quad (30b)$$

Internal quick-transition regions and their handling can be illustrated by example problem 5g:

$$\left. \begin{aligned} \epsilon f'' + 2ff' &= 0 \\ f(-\infty) &\sim -1, \quad f(\infty) \sim 1 \end{aligned} \right\} \quad (31)$$

with the solution

$$f = \tanh(y/\epsilon) \quad (32)$$

and by example problem 5h:

$$\left. \begin{aligned} \epsilon(y + \epsilon)^2 f'' + 2(f - \epsilon)[(y + \epsilon)f' - f] &= 0 \\ f'(-\infty) &\sim -1, \quad f'(\infty) \sim 1 \end{aligned} \right\} \quad (33)$$

with the solution

$$f = (y + \epsilon) \tanh(y/\epsilon) \quad (34a)$$

and the displacements (to be defined):

$$\delta^+ = \delta^- = -\epsilon \quad (34b)$$

The latter problem can illustrate the calculation of displacement and its role in matching concerning a model variable (f') which is analogous to a variable that may overshoot and then relax in a shock wave.

A class of problems closely related to these previous classes may be denoted by class 6, the problems in which statements Ib, IIId, and IIIId apply. Example problem 6:

$$\left. \begin{aligned} \epsilon f'' + f &= 1 \\ f(0) = f'(0) &= 1 \end{aligned} \right\} \quad (35)$$

with the solution

$$f = e^{1/2} \sin \left(\frac{y}{\epsilon^{1/2}} \right) + 1 \quad (36)$$

illustrates this class (see ref. 1, p. 213). Note that f has an essential singularity on $\epsilon = 0$, but not of the exponential type, and that $f' = \cos(y/\epsilon^{1/2})$ has no limit as $\epsilon \rightarrow 0$. One does not expect to find physical problems in this category.

A great deal can be learned about the various classes by studying the above simple problems. The discussion is limited here because much information is contained in the definition of each class, and the classes and example problems will be discussed later.

In attempting to solve a given problem by asymptotic or perturbation expansion, one may encounter a nonuniformity such as described by one of the properties in II. One cannot be certain in advance whether the expansion solutions to be determined will be analytic with respect to some form of the small parameter. The approach often used in inner and outer expansions is to start by assuming a power-series form of the expansion and trying to determine the respective terms. In addition, one would like to know which statement in III applies, in particular, whether the outer expansion is approached termwise algebraically or exponentially by the appropriate inner expansion. If there is no discontinuity or lost condition at a boundary (IIa), one can generally be sure that there is no quick-transition region and hence no exponential-type essential singularity (IIIa). Then, if the terms of the outer expansion are singular at $y = 0$, the approach will be algebraic (to be explained in the subsection The Matching Principle). If the nonuniformity at $\epsilon = 0$ is due to an exponential-type essential singularity, the order of the differential equation is reduced, a condition may be lost, and then, if there are no negative powers of y in the outer expansion, an appropriate inner expansion can approach termwise exponentially to a form of the outer expansion, to be shown. Thus, it will be seen (subsection entitled The Matching Principle) that statement IIIc follows from IIc. First- and higher-order boundary-layer problems in the fluid-flow theory governed by the Navier-Stokes equations are generally of this type. It is in these problems of class 5 where the procedures to be outlined below (p. 16 ff) are most advantageously used.

ESSENTIAL FEATURES OF THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS (INNER AND OUTER EXPANSIONS)

The purpose of this section is to review briefly the principal ideas of the general inner- and outer-expansion method, which is the primary basis for the new approach to be discussed in the next section. For a more complete description of the general method the reader should refer to references 1 and 21 to 25.

It is useful to introduce first the most basic principles of asymptotic expansions along with the appropriate and more or less conventional notation,

for which we may refer to Erdélyi (ref. 32). An asymptotic expansion to N terms for a function f of a small quantity ϵ and possibly of some independent variables x and y , for example, may have the form

$$\left. \begin{aligned} f(x,y,\epsilon) &= \sum_{n=1}^N f_n(x,y)\varphi_n(\epsilon) + o[\varphi_N(\epsilon)] \\ \text{as } \epsilon \rightarrow 0 \text{ with } x,y \text{ fixed} \end{aligned} \right\} \quad (37)$$

where the "smaller order" symbol $o(\)$ means that, if $A = o(B)$ as $\epsilon \rightarrow 0$, then $\lim_{\epsilon \rightarrow 0} A/B = 0$. It should also be specified that $\varphi_1 = 1$ and that $\varphi_{n+1}(\epsilon) = o[\varphi_n(\epsilon)]$ as $\epsilon \rightarrow 0$. If f can be expanded (as in eq. (37)) to N terms, then one may also write

$$\left. \begin{aligned} f(x,y,\epsilon) &= \sum_{n=1}^M f_n(x,y)\varphi_n(\epsilon) + O[\varphi_{M+1}(\epsilon)] \\ \text{as } \epsilon \rightarrow 0 \text{ with } x,y \text{ fixed} \end{aligned} \right\} \quad (38)$$

where M is any integer from 1 to $N - 1$ (ref. 32), and the order symbol $O(\)$ means that if $A = O(B)$ as $\epsilon \rightarrow 0$ (where $B \neq 0$), then $\lim_{\epsilon \rightarrow 0} A/B$ is bounded.

An asymptotic expansion for f to N terms may also be written without order symbols by using the symbol which represents "asymptotically equal." Thus,

$$\left. \begin{aligned} f(x,y,\epsilon) &\sim \sum_{n=1}^N f_n(x,y)\varphi_n(\epsilon) \\ \text{as } \epsilon \rightarrow 0 \text{ with } x,y \text{ fixed} \end{aligned} \right\} \quad (39)$$

The number N may be infinity, but the symbol \sim is still used to indicate that f may contain terms of smaller order than any quantity in the sequence of φ_n as $\epsilon \rightarrow 0$; for example, $e^{-1/\epsilon}$ is smaller than $\varphi_n = \epsilon^n$ for any integer n as $\epsilon \rightarrow 0$.

In using the method of matched asymptotic expansions, one usually starts with a power series expansion in what appears to be the logical form of the small parameter ϵ , and then modifies it as necessary if and when difficulties appear. One may also leave the form of the expansions initially unspecified so that the expansions can be constructed as the problem proceeds (ref. 1).

The first step in solving a problem for $f(x,y,\epsilon)$ by this method is to represent the solution function by the asymptotic expansion (39) where $\varphi_1 = 1$ and $N \rightarrow \infty$. Equation (39) is then substituted into the differential equation and boundary conditions of the problem. The first-order problem (for $f_1(x,y)$) may be extracted by letting ϵ become zero. One then observes that $f_1(x,y)$ is nonuniform at some values of x and y . Equation (39) is then denoted as

the outer expansion. "Inner variables" are defined by magnifying the physical variables in such a way that the inner dependent variables remain uniformly valid as $\epsilon \rightarrow 0$ and as the location of the nonuniformity of the outer problem is approached simultaneously. The inner variables are thus required to be $O(1)$ as $\epsilon \rightarrow 0$. If the nonuniformity is at $y = 0$ when $\epsilon = 0$, then the independent variable y is magnified by some function of ϵ as

$$y = \sigma_1(\epsilon)Y \quad (40)$$

The dependent variable f should be similarly magnified:

$$f(x, y, \epsilon) = \sigma_2(\epsilon)F(x, Y, \epsilon) \quad (41)$$

where the inner dependent variable F is $O(1)$ as $\epsilon \rightarrow 0$ with x and Y fixed. The functions σ_1 and σ_2 must be determined according to the requirements of the specific problem upon substituting equations (40) and (41) into the governing equations and boundary conditions.

An "inner" asymptotic expansion for F , valid in the inner region, is also left unspecified a priori and may be written as

$$\left. \begin{aligned} F(x, Y, \epsilon) &\sim \sum_{n=1}^{\infty} F_n(x, Y) \Phi_n(\epsilon) \\ \text{as } \epsilon \rightarrow 0 \text{ with } x, Y \text{ fixed} \end{aligned} \right\} \quad (42)$$

where $\Phi_1 = 1$ and $\Phi_{n+1} = o(\Phi_n)$ as $\epsilon \rightarrow 0$. As discussed above, one may try letting the sequence of $\Phi_n(\epsilon)$ be a power sequence of what appears to be the appropriate form of ϵ , until difficulty is encountered, but, in general, the form of each $\Phi_n(\epsilon)$ must be determined as one proceeds with the problem.

The solutions may be obtained for the respective terms of the inner and outer expansions when sufficient boundary conditions for each of the problems are known. Sufficient boundary conditions are provided by a procedure for matching the inner and outer expansions to each other according to the principles set forth by Kaplun (ref. 21) (see discussion by Lagerstrom, ref. 25). In essence, the outer expansion of the inner expansion matches the inner expansion of the outer expansion to appropriate orders for the respective terms. The matching principle also can often be used to determine the inner and outer asymptotic sequences of functions of ϵ (ref. 1).

The matching principle is conveniently stated with the following notation: Let f^{mo} be the m term outer expansion for f , that is,

$$f^{mo} = f_1 + \phi_2(\epsilon)f_2 + \dots + \phi_m(\epsilon)f_m \quad (43)$$

and let f^{ni} be the product of $\sigma_2(\epsilon)$ times the n term expansion of F , that is,

$$f^{ni} = \sigma_2[F_1 + \Phi_2(\epsilon)F_2 + \dots + \Phi_n(\epsilon)F_n] \quad (44)$$

If we now substitute inner variables into the outer expansion (43), we have "the m term outer expansion in inner variables," denoted as $(f^{mo})^i$; if we substitute outer variables into the inner expansion (44), we have "the n term inner expansion in outer variables," denoted as $(f^{ni})^o$. Then the first n terms of $(f^{mo})^i$ constitute the " n term inner expansion of the m term outer expansion," denoted as $(f^{mo})^{ni}$, and the first m terms of $(f^{ni})^o$ constitute the " m term outer expansion of the n term inner expansion," denoted as $(f^{ni})^{mo}$. The principle that determines the matching between the inner and outer expansions is then

$$(f^{mo})^{ni} = (f^{ni})^{mo} \quad (45)$$

The use of intermediate expansions and intermediate matching is also often necessary or convenient (see ref. 1 or 23).

After the inner and outer problems have been solved it may be desired to combine the inner and outer solutions into one solution called the "composite solution," which approaches the outer solution in the outer limit and approaches the inner solution in the inner limit. The composite solution is therefore uniformly valid over the entire region of interest. In the above notation, the "additive rule" for forming the composite solution is

$$f^c = f^{ni} + f^{mo} - (f^{mo})^{ni} \quad (46)$$

METHOD OF ASYMPTOTIC EXPANSIONS WITH DISPLACEMENT VARIABLES

In this section will be outlined a new approach to obtaining inner and outer asymptotic-expansion solutions in certain classes of problems that have been formulated in terms of differential equations and boundary conditions and that contain a small parameter. The essential features distinguishing this approach from the conventional inner- and outer-expansion method are:

- (a) a specific definition of outer and inner functions as formal sums of series that converge in some region for sufficiently small values of the expansion parameter and which are asymptotic to the exact solution;
- (b) the use of displacement variables both in defining the outer solution function and in matching the inner solution to the outer solution;
- (c) the development and statement of a matching principle that explicitly retains all displacement effects in matching inner solutions to outer solutions in a class of problems; and
- (d) a statement of certain rules often found to be useful in obtaining the appropriate forms of the expansions and magnifying factors. Application of the approach formulated here is restricted to certain classes of problems (to be discussed).

To show how this method takes advantage of the special properties of class 5 (as defined in the section "Characteristics of Several Classes of Singular Perturbation Problems") and to show the distinctions from other classes, the solution of problems in classes 1 and 5 will be discussed.

Analytic Inner and Outer Functions as Sums of Assumed Power Series

Sums of asymptotic series and of analytic expansions. - Consider a function $f(x, y, \epsilon)$ with the asymptotic expansion:

$$f(x, y, \epsilon) \sim \sum_{n=1}^{\infty} \varphi_n(\epsilon) f_n(x, y) \quad \text{as } \epsilon \rightarrow 0 \quad (47)$$

where

$$\varphi_1 = 1 \text{ and } \varphi_n(\epsilon) = o[\varphi_{n-1}(\epsilon)] \quad \text{as } \epsilon \rightarrow 0 \quad (48)$$

and where, for each $\varphi_n(\epsilon)$, there is some $m > 0$ such that

$$\epsilon^m = o[\varphi_n(\epsilon)] \quad \text{as } \epsilon \rightarrow 0$$

The series on the right side of (47) is an asymptotic series. The series may be convergent or divergent (ref. 32, p. 12). Denote by $D^*(x, y)$ the domain of all values of x, y for which the series converges for sufficiently small $\epsilon > 0$. Define $f^*(x, y)$ (in D^*) as the sum of the infinite series of functions $\varphi_n(\epsilon) f_n(x, y)$, excluding all parts of f that are $o[\varphi_n(\epsilon)]$ as $\epsilon \rightarrow 0$ for all n (e.g., terms that are exponentially small if the φ_n are powers of ϵ , $\varphi_n = \epsilon^n$):

$$f^*(x, y, \epsilon) \equiv \sum_{n=1}^{\infty} \varphi_n(\epsilon) f_n(x, y) \quad \text{in } D^*(x, y) \quad (49)$$

Although the expansion (47) does not uniquely determine its asymptotic sum f (ref. 32, p. 14) because f may contain terms of smaller order as $\epsilon \rightarrow 0$ than any $\varphi_n(\epsilon)$ of the asymptotic sequence, the sum of the series, f^* , is here defined to contain only the terms $\varphi_n f_n$ and so is uniquely defined by equation (49), if the series converges.

From this point on, only continuous functions that possess asymptotic expansions in series of powers of some function ϵ_c of the small parameter will be considered (except in the subsection "Extension of the Method to a Class of Problems With Outer and Inner Functions Nonanalytic at $\epsilon = 0$," pp. 33-36); thus it is assumed that f has the asymptotic expansion

$$f(x, y, \epsilon) \sim \sum_{n=1}^{\infty} \epsilon_c^{n-1} f_n(x, y) \quad \text{as } \epsilon \rightarrow 0 \quad (50a)$$

where

$$\epsilon_c = \epsilon_c(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0 \quad (50b)$$

and where $\epsilon_c(\epsilon)$ is determined in some way. (The subscript c does not yet have any special meaning here, but is convenient for later use.) For all x, y in $D^*(x, y)$, defined now as the domain where the series in (50a) converges for sufficiently small $\epsilon > 0$, the sum of the series excluding terms of f that are of order smaller than ϵ_c^m for any m is an analytic function of ϵ_c at $\epsilon = 0$, denoted as

$$f^*(x, y, \epsilon) \equiv \sum_{n=1}^{\infty} \epsilon_c^{n-1} f_n(x, y) \quad \text{in } D^*(x, y) \quad (50c)$$

(from the theory of analytic functions of a complex variable). Conversely, if, in some domain $D^*(x, y)$, f^* is analytic with respect to ϵ_c at $\epsilon = 0$, the series (50c) converges for sufficiently small ϵ there and is identical to the Maclaurin series in ϵ_c .

Denote the difference between f and f^* by $q^*(x, y, \epsilon)$:

$$f(x, y, \epsilon) \equiv f^*(x, y, \epsilon) + q^*(x, y, \epsilon) \quad (50d)$$

Since q^* contains only the parts of f that are smaller than any power of ϵ_c as $\epsilon \rightarrow 0$ (by the definition of f^*), then

$$q^*(x, y, \epsilon) \sim 0 + \text{exp} \quad \text{as } \epsilon \rightarrow 0 \quad (50e)$$

where "exp as $\epsilon \rightarrow 0$ " indicates terms that are exponentially small as $\epsilon \rightarrow 0$ with the other arguments (x, y) fixed.

The analytic function $f^*(x, y, \epsilon)$ is defined in a nonvanishing domain $D^*(x, y)$ if, as assumed, f has the asymptotic power-series form, (50a), and if also at some boundary, say $y_b(x_b)$, a boundary condition is applied to f that is an analytic function of ϵ_c at $\epsilon = 0$:

$$f(x_b, y_b, \epsilon) = f^*(x_b, y_b, \epsilon) = \sum_{n=1}^{\infty} \epsilon_c^{n-1} f_n(x_b, y_b) \quad (50f)$$

(The above statement is easily proved with the comparison test for convergence, with the geometric series as majorant.)

An example that conveniently illustrates the above discussion is the function

$$f(y, \epsilon) = \int_0^y \frac{e^{-t} dt}{1 + \epsilon t} + e^{-y/\epsilon} + e^{-1/\epsilon}, \quad (y \geq 0) \quad (51a)$$

$$= f^*(y, \epsilon) + q^*(y, \epsilon), \quad (0 \leq y < \infty) \quad (51b)$$

where

$$f^*(y, \epsilon) = \int_0^y \frac{e^{-t} dt}{1 + \epsilon t} = \sum_{n=0}^{\infty} (-1)^n n! \epsilon^n \left[1 - \left(1 + y + \frac{y^2}{2!} + \dots + \frac{y^n}{n!} \right) e^{-y} \right] \quad (51c)$$

and

$$q^*(y, \epsilon) = e^{-y/\epsilon} + e^{-1/\epsilon} \quad (51d)$$

The function $f(y, \epsilon)$ is not analytic with respect to ϵ at $\epsilon = 0$, but its asymptotic expansion for small ϵ is the asymptotic power series in (51c). The series (51c) converges for $|\epsilon| < |1/y|$, and so f^* is analytic with respect to ϵ at $\epsilon = 0$ for all finite y , and $D^*(y)$ is then $0 \leq y < \infty$. For $y = \infty$, the function $\int_0^y (1 + \epsilon t)^{-1} e^{-t} dt$ becomes nonanalytic at $\epsilon = 0$, since the radius of convergence about $\epsilon = 0$ in the complex ϵ plane vanishes, and the asymptotic expansion

$$f(\infty, \epsilon) \sim \int_0^{\infty} \frac{e^{-t} dt}{1 + \epsilon t} \sim 1 - 1!\epsilon + 2!\epsilon^2 - 3!\epsilon^3 + \dots \quad (51e)$$

diverges (cf. ref. 32, pp. 1-3). It is pertinent to note here that, in this typical example, f^* is not a good approximation to f very near $y = 0$ for small ϵ , since at $y = 0$, $f = 1 + e^{-1/\epsilon}$ and $f^* = 0$. Now let $y \equiv \epsilon Y$ and consider the same function $f(y, \epsilon)$ as in equation (51a) but with y replaced by ϵY :

$$f(\epsilon Y, \epsilon) = \int_0^{\epsilon Y} \frac{e^{-t} dt}{1 + \epsilon t} + e^{-Y} + e^{-1/\epsilon}, \quad (Y \geq 0) \quad (52a)$$

This function of Y and ϵ , which is identically equal to $f(y, \epsilon)$, is also nonanalytic with respect to ϵ at $\epsilon = 0$, but has the asymptotic expansion

$$f(\epsilon Y, \epsilon) \sim e^{-Y} + \sum_{j=0}^{\infty} A_j(Y) \epsilon^{2j+1} + \sum_{j=0}^{\infty} B_j(Y) \epsilon^{2j+2} \quad (52b)$$

where:

$$A_j(Y) = \sum_{k=0}^j \frac{(-1)^k k! Y^{2j+1-k}}{(2j+1-k)!} ; \quad B_j(Y) = \sum_{k=0}^j \frac{(-1)^k k! Y^{2j+2-k}}{(2j+2-k)!} \quad (52c)$$

The two series in (52b) converge for all $|\epsilon^2| < |1/Y|$. Thus, $f(\epsilon Y, \epsilon)$ can be written, by analogy with equation (50d), as

$$f(\epsilon Y, \epsilon) = F(Y, \epsilon) + Q(Y, \epsilon) , \quad (0 \leq Y < \infty) \quad (53a)$$

where $F(Y, \epsilon)$ is the analytic function in ϵ at $\epsilon = 0$ which is the sum of the power series in ϵ , containing all parts of f not exponentially small as $\epsilon \rightarrow 0$ with Y fixed:

$$F(Y, \epsilon) = \int_0^{\epsilon Y} \frac{e^{-t} dt}{1 + \epsilon t} + e^{-Y} \quad (53b)$$

and where

$$Q(Y, \epsilon) = e^{-1/\epsilon} \sim 0 + \exp \quad \text{as } \epsilon \rightarrow 0 \text{ with } Y \text{ fixed} \quad (53c)$$

The domain of definition of $F(Y, \epsilon)$ (like $f^*(y, \epsilon)$) is $(0 \leq Y < \infty)$. The function $F(Y, \epsilon)$ is a good approximation to $f(y, \epsilon)$ near $y = 0$ for small ϵ .

In another example (cf. example problem 5g, eqs. (31)),

$$f = \tanh(x/\epsilon) \quad (54a)$$

has the asymptotic expansions:

$$\left. \begin{aligned} f &\sim 1 + \exp & \text{as } \epsilon \rightarrow 0 \ (x > 0) \\ f &\sim -1 + \exp & \text{as } \epsilon \rightarrow 0 \ (x < 0) \end{aligned} \right\} \quad (54b)$$

In this case, two asymptotic analytic functions f^* are needed:

$$f^{*+} = 1 , \quad x > 0 ; \quad f^{*-} = -1 , \quad x < 0 \quad (54c)$$

since the asymptotic solution is discontinuous at $x = 0$. This is an example of Stokes' phenomenon (ref. 4). For this case, the analytic function of ϵ that is uniformly valid in the region of the nonuniformity of $f(x, \epsilon)$ is

$$F(X) = \tanh X \quad (54d)$$

where $X = x/\epsilon$ is held fixed as $\epsilon \rightarrow 0$.

In solving for an unknown function f in a singular perturbation problem, it is not known for certain whether asymptotic expansions of the type (50a) can properly represent the function. If one assumes that such an expansion does represent the function, then one can speak of the analytic sum f^*

of the assumed power series (or, more briefly, assumed-analytic function). If the form (50a) is valid as assumed, then the function f^* defined by equation (50c) exists at least in some neighborhood D^* of a boundary where a condition is applied to f that is analytic with respect to ϵ_c at $\epsilon = 0$.

In the approach to finding inner and outer expansions used here, assumed-analytic functions (like f^* and F in equations (51) to (53)) will be used for both the outer and magnified inner functions. The outer function is assumed to be a power series in some parameter $\epsilon_o = \epsilon_o(\epsilon)$ and the magnified inner function to be a power series in $\epsilon_i = \epsilon_i(\epsilon)$. These power series are assumed to represent the solution asymptotically as $\epsilon \rightarrow 0$ and are analytic functions of ϵ_o and ϵ_i , respectively, at $\epsilon = 0$ in the respective regions where the series converge. If consistent solutions cannot be determined for the respective terms of an expansion, then the assumed-power-series function is not applicable, that is, cannot represent the exact solution function asymptotically. If the various terms in the assumed expansion can be determined and are consistent and noncontradictory, then it is assumed that the function f^* given by the expansion represents the function f asymptotically. This is equivalent to an assumption of uniqueness of the asymptotic expansion, to the order in ϵ_c considered. The asymptotic expansion of a function is known to be unique for a given asymptotic sequence (ref. 32, p. 13). If, in a given problem, there is reason to doubt that outer and magnified inner functions that are analytic in some $\epsilon_c(\epsilon)$ at $\epsilon = 0$ can be defined even though the first few terms of a power series in ϵ_c are found, then a slightly different point of view may be adopted (see subsection "Extension of the Method . . .," pp. 33-36), but the terms already found remain valid to the order considered.

In many problems either the outer function or the inner function or both may be equivalent to the exact solution function. For instance, example problem 1 will have both outer and inner functions equivalent to the exact solution function; in class 5, however, the outer solution cannot be equivalent to the exact solution, but in all the example problems of class 5 except 5f the inner solution function is the exact solution. In example problem 5f neither the outer function nor the inner is the exact solution.

Definitions of inner and outer functions.— The definitions of the outer and inner functions to be used in the present approach are contained in the following paragraphs.

Assume that $f(x, y, \epsilon)$ can be separated into two parts, one part (f^o) which is analytic in some region with respect to some form $\epsilon_o(\epsilon)$ at $\epsilon = 0$ and the other (q^o) decreasing to zero faster than any power of ϵ_o :

$$f(x, y, \epsilon) \equiv f^o(x, y, \epsilon) + q^o(x, y, \epsilon) \quad (55a)$$

where

$$\left. \begin{aligned} f^o & \text{ is analytic with respect to } \epsilon_o = \epsilon_o(\epsilon) \quad \text{at } \epsilon = 0 \text{ in some } D^o(x, y); \\ q^o & \sim 0 + \exp \quad \text{as } \epsilon \rightarrow 0 \text{ with } x, y \text{ fixed} \end{aligned} \right\} \quad (55b)$$

Note that only terms contributing to f^0 will appear in any asymptotic power-series expansion of f as $\epsilon \rightarrow 0$ with x, y fixed. Except near some nonuniformity of f at some (x, y) , f^0 is then a valid representation of f and is defined as the outer function (if, in fact, f has an asymptotic power series).

If the order of the differential equation is not reduced and all the boundary conditions can be satisfied, one can treat the problem as class 1. In this class there are no exponentially small terms in the exact solution, so $q^0 \equiv 0$ and f^0 and its expansion for small ϵ are equivalent to f :

$$f^0 = f, \quad (\text{class 1}) \quad (56)$$

The outer expansion is then determined as the power series in ϵ in the form of, say, equation (50c) which, by direct substitution, is made to satisfy the differential equation and all the boundary conditions on $f(x, y, \epsilon)$. This expansion may then be not uniformly valid near some x, y because individual terms are singular there.

In boundary-layer-type problems (class 5) the order of the differential equation is reduced upon letting $\epsilon \rightarrow 0$. Then the limit solution obtained is found to be discontinuous; that is, either it does not satisfy a condition of continuity of f (or one of its derivatives) at an internal location (called a Stokes line; cf. ref. 4), or it cannot satisfy (continuously) a boundary condition required of f or one of its derivatives. The discontinuous behavior of the limit function indicates the presence of terms in f that vary rapidly (exponentially) near the location where the discontinuity is encountered. The discontinuity itself is an asymptotic representation of a rapid variation of the dependent variable over a very small range of the independent variable (say over y near $y = 0$).

An outer function f^0 that corresponds to the nonuniformity at $y = 0$ in a problem of class 5 may be defined by equations (55), where q^0 contains the exponentially decaying quantities (as $\epsilon \rightarrow 0$) including those that cause the rapid variation of f and its derivatives near $y = 0$. (In the case of an internal discontinuity, the function f^0 in equations (55) is given by two different analytic functions, f^{0+} and f^{0-} , one valid on each side of the discontinuity.) Since q^0 and its derivatives are exponentially small everywhere except very near $y = 0$ when ϵ is small, f^0 is a valid asymptotic function representing f everywhere except very near $y = 0$. Then f^0 must satisfy, to within exponentially small amounts, the differential equation and all boundary conditions of f and its derivatives except those at $y = 0$. Further conditions (at or near $y = 0$) that determine f^0 are discussed below.

Denote by $w(x, y, \epsilon)$ the particular dependent variable, or derivative of f , whose limit function in the outer problem is discontinuous; also define w^0 as the corresponding function in the problem for f^0 and let

$$w(x, y, \epsilon) = w^0(x, y, \epsilon) + q_w(x, y, \epsilon) \quad (57)$$

where $q_w \sim 0 + \exp$ as $\epsilon \rightarrow 0$. (In example problem 5b the outer limit function is $f_1(y) = y$; the derivative that cannot satisfy a condition at $y = 0$

is $f_1'(y) = 1$, so, in that problem, $w = f'(y, \epsilon)$.) At $y = 0$, neither the complete function f^0 nor any of its derivatives can generally satisfy exactly the same conditions as are satisfied by f and its derivatives because of the significant terms in q^0 and its derivatives at $y = 0$. Consider example problem 5b, for which the solution f is given by equation (22a) and $f' = \text{erf}(y/\sqrt{2\epsilon})$, and in which f and its derivatives are represented asymptotically to within exponentially small quantities by the outer functions

$$f^0(y) = y - \epsilon^{1/2} \sqrt{2/\pi}, \quad f^{0'} = 1, \quad f^{0''} = 0 \quad (58a)$$

At $y = 0$:

$$\left. \begin{aligned} f &= 0, & f' &= 0, & f'' &= \epsilon^{-1/2} \sqrt{2/\pi}, & f''' &= 0, \dots \\ f^0 &= -\epsilon^{1/2} \sqrt{2/\pi}, & f^{0'} &= 1, & f^{0''} &= 0, & f^{0'''} &= 0, \dots \end{aligned} \right\} \quad (58b)$$

Note that the deviation of w^0 from $w = f'$ at $y = 0$ is $O(1)$ as $\epsilon \rightarrow 0$, whereas the deviations of the derivatives of lower order than w^0 are $o(1)$ as $\epsilon \rightarrow 0$. One possible way to regard this is that, to first order, only the condition on $w = f'$ must be omitted from the outer problem, but that, to order ϵ , the condition on f is not satisfied. But then there would not be sufficient conditions to determine f^0 to second order. A more useful approach is to omit only the condition on w from the outer problem and to require f^0 to satisfy the same condition as f (i.e., $f = 0$ in this problem) in some way, to make f^0 determined. But we see that f^0 vanishes at $y = \delta = \epsilon^{1/2} \sqrt{2/\pi}$ rather than at $y = 0$. Thus, we require the conditions on all lower order derivatives of f than w to be satisfied in the problem for f^0 , but allow those conditions each to be satisfied at an appropriate displacement surface, $y = \delta$. In general, there is a displacement associated with each condition applied to f^0 near $y = 0$. Corresponding to each condition

$$y = 0, \quad \partial^m f(x, y, \epsilon) / \partial y^m = a_m \quad (59a)$$

for all lower order derivatives than w , the displacement δ for the function $\partial^m f^0(x, y, \epsilon) / \partial y^m$ is therefore the root δ of

$$\partial^m f^0(x, \delta, \epsilon) / \partial y^m = a_m \quad (59b)$$

Since δ is small, the conditions can usually be transferred by Taylor's series to $y = 0$. The displacements are unknown a priori in solving a problem, but are determined as part of the solution (to be shown). One must be careful to note that even in problems involving only real variables with purely real solutions, the displacements may be complex. In example problem 5e, for which $w = f''(y, \epsilon)$ and for which two conditions must be applied in the problem for f^0 near $y = 0$, the condition $f^0 = 0$ is satisfied at $y = \delta_a = \epsilon(1 \pm i)$ and $f^{0'} = 0$ is satisfied at $y = \delta_b = \epsilon$. (Complex displacements in general need not be considered unless a difficulty is encountered not otherwise explainable in matching inner solutions to outer solutions or in determining some unknown

quantity.) With the above requirement that all conditions on dependent functions of lower order than w be applied at displacement surfaces, the outer function f^0 is determined in terms of unknown displacements in class 5.

Although the individual terms of the expansion of f^0 will be determined from reduced-order equations, the function f^0 is still determined by the complete differential equation. The condition on w at $y = 0$ is replaced by the condition that there be no boundary layer, or rapid transition, in the function f^0 near $y = 0$. This insures a unique solution for f^0 .

In the inner region of the nonuniformity (the vicinity of the location where the terms of the outer expansion become infinite in class 1 or the vicinity of the discontinuity of the outer solution in class 5), an asymptotic function representing the solution function is to be found by magnifying both the independent and dependent variables in such a way that the solution remains uniformly valid as $\epsilon \rightarrow 0$. Let the magnified independent variable be

$$Y = y/\sigma_1(\epsilon) \quad (60a)$$

and let $F = f^i/\sigma_2(\epsilon)$ denote the assumed-analytic function (cf. discussion of f^* above) representing (at least asymptotically) the magnified solution function, $f/\sigma_2(\epsilon)$, in the inner region, that is,

$$f(x, y, \epsilon) = f^i(x, Y, \epsilon) + q^i(x, Y, \epsilon) \quad (60b)$$

where

$$\left. \begin{aligned} F(x, Y, \epsilon) = f^i(x, Y, \epsilon)/\sigma_2(\epsilon) \text{ is assumed to be} \\ \text{analytic with respect to some quantity } \epsilon_1 = \epsilon_1(\epsilon) \\ \text{at } \epsilon = 0 \text{ in some } D^i(x, Y) ; \text{ and} \\ q^i \sim 0 + \exp \text{ as } \epsilon \rightarrow 0 \text{ with } x, Y \text{ fixed.} \end{aligned} \right\} \quad (60c)$$

All inner dependent variables ($F, \partial F/\partial Y$, etc.) are considered to be $O(1)$ as $\epsilon \rightarrow 0$ with Y held fixed. Substitution of equation (60b) into the differential equation of the problem for f and into the boundary conditions on f and its derivatives at $y = 0$ requires that $f^i(x, Y, \epsilon)$ satisfy those equations and conditions to within exponentially small amounts as $\epsilon \rightarrow 0$ with x, Y fixed. The problem for $F(x, Y, \epsilon)$ is then determined if appropriate forms for $\sigma_1(\epsilon)$ and $\sigma_2(\epsilon)$ are found and if sufficient outer boundary conditions are provided by appropriate matching of the inner function to the outer function. Some particularly useful matching relationships can be derived from the definitions of the outer and inner functions (to be shown below).

The functions $\epsilon_0(\epsilon)$, $\epsilon_1(\epsilon)$, $\sigma_1(\epsilon)$, and $\sigma_2(\epsilon)$ must be determined according to the specific problem. Some rules that are often useful for determining σ_1 and σ_2 will be discussed later. When σ_1 and σ_2 are known, and the inner problem is set up in terms of x, Y, F , and ϵ , one can assume ϵ_0 and ϵ_1 to be the forms of the small parameter occurring explicitly in the completely-specified outer and inner problems (including the conditions to be

applied at the displacement surfaces in the outer problem in class 5 and the conditions to be supplied by the matching principle in the inner problem).

For the most convenient use of the matching principle, both the outer and inner problems may be solved in terms of a single small parameter $\epsilon_c = \epsilon_c(\epsilon)$ with respect to which both ϵ_0 and ϵ_1 are analytic. For example, if $\epsilon_0 = \epsilon^{1/2}$ and $\epsilon_1 = \epsilon^{1/3}$, then the common form of the small parameter to use as the expansion parameter is $\epsilon_c = \epsilon^{1/6}$. The assumed power series are then

$$f^0(x, y, \epsilon) = f_1(x, y) + \epsilon_c f_2(x, y) + \epsilon_c^2 f_3(x, y) + \dots \quad (61)$$

$$F(x, Y, \epsilon) = F_1(x, Y) + \epsilon_c F_2(x, Y) + \epsilon_c^2 F_3(x, Y) + \dots \quad (62)$$

which will be substituted into the inner and outer problems so that the respective terms can be evaluated. In the subsection "Extension of the Method . . .," (pp. 33-36) the requirement of analyticity of f^0 and F at $\epsilon_c = 0$ is relaxed, but the terms of equations (61) and (62) remain valid if determined.

Magnifying Factors

The magnifying factors σ_1 and σ_2 in equations (60) define the relationship between the inner functions $F(x, Y, \epsilon)$ and f^1 . The order of magnitude of σ_1 represents the order of the width (measured in the units of y) of the region of nonuniformity, and σ_2 represents the order of magnitude of f in that region as $\epsilon \rightarrow 0$.

For class 1, a solution that is valid near the singularity at $y = 0$ as $\epsilon \rightarrow 0$ must be found by defining an inner function that is, in general, analytic as $y \rightarrow 0$ and $\epsilon \rightarrow 0$ according to equations (60). In the problems of class 1, the outer expansion (expansion of f^0) is the same as the formal perturbation expansion of $f(x, y, \epsilon)$. The proper magnification factors σ_1 and σ_2 are therefore determined by comparing successive terms in the outer expansion, observing how the nonuniformity "grows" (for every order of ϵ added to the expansion, a certain higher-order infinity is encountered; see ref. 1), then observing the order of the dependent variable as the singularity is approached, and defining σ_1 and σ_2 so that Y and F are of order unity in the inner region (as ϵ and $y \rightarrow 0$). The terms of the inner expansion contain constants of integration that must be evaluated by matching the inner and outer expansions. The matching principle is introduced in a later section, but it may be noted that for problems of class 1, the matching process is equivalent to that used in the conventional inner and outer expansion method except for the particular use of the common parameter ϵ_c here.

In problems of class 5, some simple rules have been found to be useful in determining the magnifying factors σ_1 and σ_2 . If these rules lead to a useful result without contradictions in a given problem, then the choices made for σ_1 and σ_2 are valid. The first rule is: If a function denoted by w is discontinuous (but bounded) in the outer problem (e.g., if w^0 cannot satisfy a boundary condition required of w as $\epsilon \rightarrow 0$ with y held fixed), then require

$$w^i = O(1) \quad \text{as } \epsilon \rightarrow 0 \quad \text{with } Y = y/\sigma, \text{ held fixed} \quad (63)$$

The function w may be f itself or some higher order y derivative of f . For example (as in example problem 5b), suppose $w^0 = df^0/dy$ cannot satisfy the boundary condition required of df/dy at $y = 0$. Then require that

$$w^i = df^i/dy = O(1) \quad \text{as } \epsilon \rightarrow 0 \quad \text{with } Y \text{ fixed}$$

Since all the magnified inner functions (F and its Y derivatives in example 5b) are required to be $O(1)$ as $\epsilon \rightarrow 0$, we can then write

$$\frac{dF}{dY} = \frac{\sigma_1}{\sigma_2} \frac{df^i}{dy} = O(1) \quad \text{as } \epsilon \rightarrow 0$$

For example problem 5b, the result is that

$$\sigma_1/\sigma_2 = O(1) \quad \text{as } \epsilon \rightarrow 0$$

and we may let $\sigma(\epsilon) = \sigma_1(\epsilon) = \sigma_2(\epsilon)$. The reasoning (for a general problem) is as follows (Refer to fig. 1.): The discontinuity of w^0 is an asymptotic

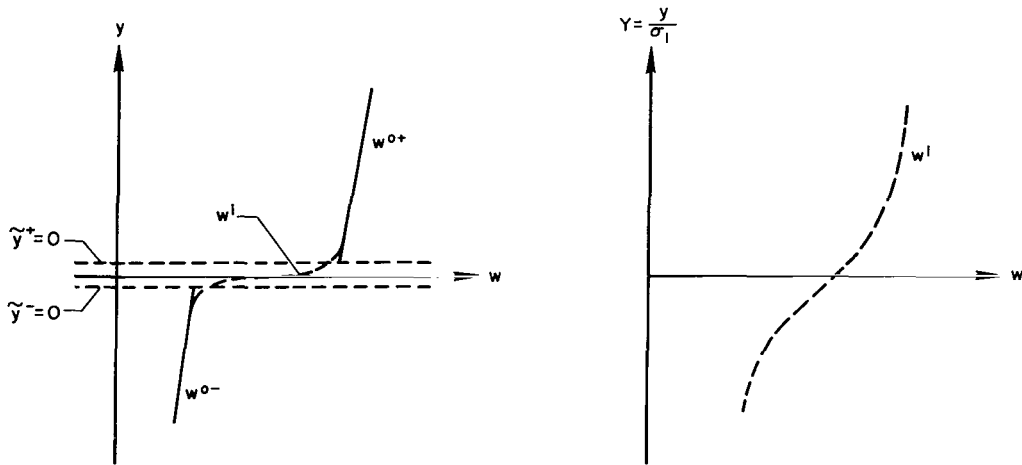


Figure 1.- Illustration of variable w that has internal discontinuity in the outer problem.

representation of a rapid variation of w . To study the variation of w in the inner region represented by that discontinuity, one must magnify the y scale ($Y = y/\sigma_1$) but keep the w scale fixed as $\epsilon \rightarrow 0$. This is so because, if the order of magnitude in the w scale were changed as $\epsilon \rightarrow 0$, no variation of w could be observed (on the right side of fig. 1) because all the values of w^i being considered would either become infinite or vanish as $\epsilon \rightarrow 0$. Hence, in the inner solution we need $w^i = O(1)$ as $\epsilon \rightarrow 0$. The type of function for which w is discontinuous at a boundary can also be referred to for the above discussion. In the same scheme as figure 1, that function would be as shown in figure 2. There remains one unknown magnifying factor, which can be found by the second rule, the principle of least degeneracy (attributable to Van Dyke (ref. 1) in the general method of matched asymptotic

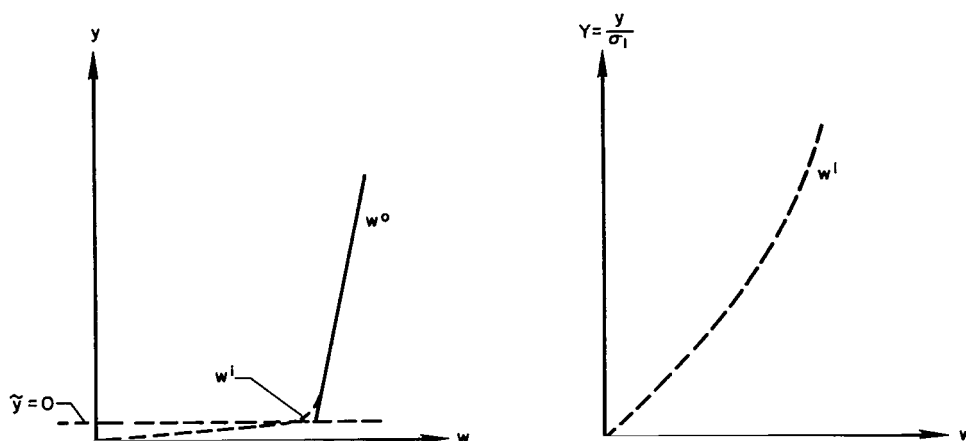


Figure 2.- Illustration of variable w that has lost boundary condition in the outer problem.

expansions). Equations (60) are substituted into the differential equation and the result is examined for the limiting process $\epsilon \rightarrow 0$. As $\epsilon \rightarrow 0$, the equation must not degenerate into a lower order differential equation and the most significant terms must not be lost in the first-order problem. (For example problem 5b, then, $\sigma = \epsilon^{1/2}$.)

In problems of class 5, the outer problem is often influenced by the inner problem, that is, there are displacement effects. Hence, ϵ_0 and ϵ_1 cannot be determined before the inner and outer problems are completely specified, including the inner boundary conditions that are not lost in the outer problem, to be applied at the displacement surfaces, and including the outer boundary conditions to be supplied to the inner problem by the matching principle. The independent displacement variable \tilde{y} (to be defined) will be seen to contain ϵ , so that ϵ then occurs in the boundary conditions of the outer problem and will also influence the inner problem through the matching process. The choice of ϵ_c therefore depends on the form of the displacement variable \tilde{y} .

Displacement Variables

The influence of the nonuniform region on the outer solution is known as the displacement effect. The most common example occurs in fluid boundary layers. The flow around a body in a uniform stream of fluid would be an inviscid flow if the body were frictionless and if there were no other source of viscous effects. If, however, the fluid flowing at high Reynolds number ($Re \rightarrow \infty$) adheres to the surface, most of the flow is inviscid but the viscous effects very near the surface produce the quick-transition region or boundary layer. The outer inviscid flow is then displaced a small distance from the body in relation to the corresponding flow over a frictionless body. There is a surface, which we denote as the displacement surface, that would be the equivalent frictionless body for the outer displaced flow. (Refer to Lighthill, ref. 33, for a discussion of displacement; also see Mangler, ref. 34.)

Any mathematical problem that contains a displacement effect (cf. example problems 5b and 5c) can be thought of as having a displacement surface (or more than one, depending on the number of inner boundary conditions). Let y be the coordinate normal to and measured from the actual surface where the nonuniformity (discontinuity or lost boundary condition) occurs and \tilde{y} (the displacement variable) be the coordinate measured from the displacement surface, whose location is unknown. The displacement is then $y - \tilde{y}$ and is, in general, a function of the other independent variables (say x and ϵ):

$$\delta \equiv \sigma_\delta(\epsilon)\Delta(x, \epsilon) \equiv y - \tilde{y} \quad (64)$$

where $\sigma_\delta(\epsilon)$ is the order of magnitude of δ as $\epsilon \rightarrow 0$. If the displacement is real, it is generally of the same order as the region of nonuniformity ($O(\sigma_1)$ as $\epsilon \rightarrow 0$), that is

$$\sigma_\delta = \sigma_1 \quad \text{for } \delta \text{ real} \quad (65)$$

(However, in the special case where δ is complex, it may have a different order of magnitude than σ_1 ; in example problem 5d, $\sigma_1 = \epsilon$ but $\delta = O(i\epsilon^{1/2})$.)

Since the outer function f^0 contains no exponentially small terms, the function $\delta(x, \epsilon)$, as the root of equation (59b), also will contain no exponentially small terms. It is then compatible with the assumption that f^0 is analytic with respect to some ϵ_c to assume Δ is also analytic with respect to ϵ_c . Thus,

$$\Delta(x, \epsilon) = \Delta_1(x) + \epsilon_c \Delta_2(x) + \epsilon_c^2 \Delta_3(x) + \dots \quad (66)$$

(where, indeed, this requirement is to be accounted for in determining ϵ_c). The terms of Δ are to be determined asymptotically by the inner solution $F(x, Y, \epsilon)$.

Useful relationships between the outer displacement variables and the inner variables are:

$$\tilde{y} = \sigma_1 Y - \sigma_\delta [\Delta_1(x) + \epsilon_c \Delta_2(x) + \epsilon_c^2 \Delta_3(x) + \dots] \quad (67a)$$

$$\tilde{x} = x \quad (67b)$$

$$Y = \tilde{y}/\sigma_1 + (\sigma_\delta/\sigma_1) [\Delta_1(\tilde{x}) + \epsilon_c \Delta_2(\tilde{x}) + \epsilon_c^2 \Delta_3(\tilde{x}) + \dots] \quad (67c)$$

For use of displacement variables in matching, define an outer displacement function \tilde{f}^0 , which is a function of \tilde{x} , \tilde{y} , and ϵ and is given by

$$\tilde{f}^0(\tilde{x}, \tilde{y}, \epsilon) = f^0(x, y, \epsilon) \quad (68)$$

and its expansion by

$$\tilde{f}^0 = \tilde{f}_1(\tilde{x}, \tilde{y}) + \epsilon_c \tilde{f}_2(\tilde{x}, \tilde{y}) + \epsilon_c^2 \tilde{f}_3(\tilde{x}, \tilde{y}) + \dots \quad (69)$$

Note that equations (68) and (64) together, along with $\tilde{x} = x$, simply represent a transformation of variables, that the power-series expansion (69)

is equivalent to (61) (because of eq. (68)), but that generally the \tilde{f}_n are not the same functions of \tilde{x}, \tilde{y} as the f_n are functions of x, y (except, of course, for the first term, \tilde{f}_1). Naturally, n terms of the expansion of f^0 agree with n terms of \tilde{f}^0 to the appropriate order (i.e., to $O(\epsilon_c^{n-1})$). Note also that a set of displacement variables corresponding to each inner boundary condition must be applied to f^0 . In a large class of problems, however, there is only one displacement. In that case, the entire outer problem may be transformed to displacement variables (\tilde{f}^0, \tilde{x} , and \tilde{y}) before solving it, or it may be solved in terms of x and y , then transformed to displacement variables (see application to blunt-body fluid flow problem below).

The Matching Principle

Outer boundary conditions to be satisfied by the respective terms of the inner expansion must be obtained by matching the expansion of the inner function term by term to the expansion of the outer function. The matching referred to here is an analytical matching to obtain rigorously the form of each term in an expansion, in contrast to frequently used approximate methods whereby functions or expansions are "matched" (or "patched") at some arbitrary point (cf. discussion by Van Dyke, ref. 1, p. 89, and Chang, ref. 35, pp. 819-820). The matching principle (primarily due to Kaplun) in the conventional inner and outer expansion method is expressed by equation (45). The displacement variables to be used here are not employed there and the displacement effects do not enter into the process of matching. The following paragraphs discuss several proposed forms of the matching principle to apply in the present approach to constructing inner and outer expansions. These proposed forms are similar to Kaplun's proposition. It has been found that, in problems of class 5, a form of the matching principle can be applied that retains explicitly all displacement effects. The nature of the displacement effects, as well as their role in the relationship between the inner and outer solutions, is then more easily and completely determined than in the conventional method.

For this discussion of matching, let superscript i denote a function that has been transformed from outer to inner (magnified) variables and superscript o denote a function transformed from inner to outer variables, in addition to the previous use of f^o and f^i to denote outer and inner functions. Then $(f^o)^i$ is the entire outer function transformed to inner variables. A combination of equation (55a) written in inner variables and equation (60b) gives

$$f^i = (f^o)^i + (q^o)^i - q^i \quad (70)$$

The difference $(q^o)^i - q^i$ consists only of the terms of q^o that, when transformed to inner variables, become of larger order than $O(\epsilon^m)$, where m is some finite number, as $\epsilon \rightarrow 0$; but those terms are then exponentially small as $Y \rightarrow \infty$. (In the example of equations (51) to (53), $q^i - (q^o)^i = -e^{-Y}$.) Thus,

$$f^i \sim (f^o)^i + \exp \quad \text{as } Y \rightarrow \infty \quad (71)$$

A direct corollary of (71) is that, to order $\sigma_2 \epsilon_c^{n-1}$ in inner variables,

$$f^{ni} \sim (f^0)^{ni} + \exp \quad \text{as } Y \rightarrow \infty \quad (72)$$

where superscript ni indicates n terms of an expansion written in inner variables. Note that for class 1 $q^1 = q^0 = 0$ and there is no difference between f^1 and $(f^0)^1$. (In class 1 the only difference is in the expansions.) The asymptotic relations (71) and (72) are valid in regard to any function f for which f^0 and f^1 can be defined by equations (55) and (60). Equations (71) and (72) are useful in themselves only if the complete unexpanded outer function (or its form) is known. However, generally useful forms of the asymptotic relation for the termwise matching of expansion solutions can be deduced directly from the form (72) (as seen below).

In the present matching procedure, when an outer expansion is to be matched to an inner expansion, the outer will always be written in terms of the displacement variables \tilde{x}, \tilde{y} . The superscript "no" will indicate n terms of the expansion of $\tilde{f}^0(\tilde{x}, \tilde{y}, \epsilon)$ (see eqs. (68) and (69)). Thus, the n term expansions to be matched are:

$$f^{no} = \tilde{f}_1(\tilde{x}, \tilde{y}) + \epsilon_c \tilde{f}_2(\tilde{x}, \tilde{y}) + \epsilon_c^2 \tilde{f}_3(\tilde{x}, \tilde{y}) + \dots + \epsilon_c^{n-1} \tilde{f}_n(\tilde{x}, \tilde{y}) \quad (73)$$

$$f^{ni} = \sigma_2(\epsilon)[F_1(x, Y) + \epsilon_c F_2(x, Y) + \dots + \epsilon_c^{n-1} F_n(x, Y)] \quad (74)$$

If the transformation by equations (67a) and (67b) is used in (73), and if the transformation by equations (67b) and (67c) is substituted into (74), one obtains $(f^{no})^1$ and $(f^{ni})^0$, respectively. If these expansions are then truncated to n terms (i.e., to order ϵ_c^{n-1} in outer variables and to order $\sigma_2 \epsilon_c^{n-1}$ in inner variables), one has $(f^{no})^{ni}$ and $(f^{ni})^{no}$.

If the first n terms (to order ϵ_c^{n-1}) of an expansion of f^0 contain all the terms of f^0 that would appear in the first n terms in inner variables, $(f^0)^{ni}$, then the quantity f^0 in (72) can be replaced by f^{no} . This condition is apparently satisfied in problems of class 5 if all of the following (to be discussed) are true:

- | | | |
|---|---|------|
| <p>(a) the outer expansion is first written in displacement variables (eq. (73));</p> <p>(b) there is only one displacement in the function being matched; and</p> <p>(c) $\delta = O(\sigma_1)$ as $\epsilon \rightarrow 0$.</p> | } | (75) |
|---|---|------|

Then

$$f^{ni} \sim (f^{no})^{ni} + \exp \quad \text{as } Y \rightarrow \infty, \text{ (class 5)} \quad (76)$$

In class 5 (as defined in pp. 4-13), there are no negative powers of y in the expansions of the f_n for small y . Then the terms of f^0 that are $O(\epsilon_c^{n-1})$, when transformed to inner variables, will become at least as small as

order $\sigma_2 \epsilon_c^{n-1}$ for each n . Then, if (75a,b,c) are satisfied, the terms of the displacement in the outer solution that will affect the n th-order inner problem are not lost upon truncation of the outer expansion at n terms and the terms of the displacement that affect the n th inner problem are also carried along in the transformation of f^{no} by equation (67a) to inner variables. (These statements are easily checked in any given problem.) The respective terms of the expansion of the displacement can then be evaluated in the inner solution to satisfy (76) exponentially. In example problem 5b, $f^0(y, \epsilon) = y - \epsilon^{1/2} \Delta_1$ and $\tilde{f}^0 = \tilde{y} = f^{10}$. Thus, f^{10} contains the first-order displacement (the entire displacement in this case). Transformation to inner variables then gives $(f^{10})^{11} = \epsilon^{1/2}(Y - \Delta_1)$, which carries along the first-order displacement, to be evaluated by compatibility with the first inner solution. If (75b) is not satisfied, that is, if f^0 contains more than one displacement in a given problem (because of more than one displaced boundary condition to be satisfied by the outer function), then all the n th order displacements cannot in general be carried through the above process without some loss. However, if the terms of only one displacement (expanded in ϵ_c) appear in an expansion of some derivative or function, then the matching rule can be used for that function using the appropriate displacement variables. (This occurs in example problem 5e.) If (75c) is not satisfied (i.e., if δ is larger than $O(\sigma_1)$ as $\epsilon \rightarrow 0$), which is possible if δ is complex (cf. example problem 5d), then it may not appear possible to satisfy (76). (In problems where this occurs, the matching rule in the conventional method of matched asymptotic expansions also does not work unless one resorts to finding an intermediate expansion (cf. ref. 1). In the present method, although the relation (72) is valid and could be used if δ were known completely, the relation (76) in that precise form may not work if δ is larger than σ_1 because significant terms do not appear in f^{no} that must appear in the asymptotic limit of f^{ni} , and f^{ni} can then not match asymptotically to $(f^{no})^{ni}$. The correct order of magnitude of δ , and consequently the proper form to replace f^{no} in (76), can then be determined by using the concept of the intermediate expansion. A displaced intermediate-expansion solution replaces f^{no} in the matching rule and is determined by setting $y = \epsilon^\alpha \hat{y} - \delta$, $f^0 = \epsilon^\beta \hat{f}$; specifying β in terms of α so that w is $O(1)$ in the intermediate region; and determining α so that $\sigma_\delta = \epsilon^\alpha$ (where $\delta = \sigma_\delta \Delta$) and so as to make possible the matching of the intermediate displacement expansion with the inner solution. Since $y = O(\delta)$ in the "overlap" region where the matching is to be accomplished and for which an intermediate expansion is needed, this procedure makes $\Delta = O(1)$ in " \hat{y} -space" in the intermediate problem and determines the \hat{y} -space where the overlap is possible. Then the large complex displacement, which caused the need for the intermediate expansion for matching, is determined in the matching process. In example problem 5d, where $\sigma_1 = \epsilon$ and $w = df/dy$, it is found that $\beta = \alpha = 1/2$.

In classes of problems where the approach of the individual terms of the inner solution to the respective terms of the outer solution is not necessarily exponential, the asymptotic relation (72) can be replaced only by

$$f^{ni} \sim (f^{no})^{ni} + \text{smaller alg} \quad \text{as } Y \rightarrow \infty \quad (77)$$

where $Y = 0$ is the locus of the nonuniformity and where "smaller alg as $Y \rightarrow \infty$ " means algebraically decreasing terms that are smaller as $Y \rightarrow \infty$ than

those contained in $(f^{no})^{ni}$. The appearance of the algebraically decaying terms in (77) (e.g., for class 1), as opposed to the exponential approach in class 5, is caused by the singularity at $y = 0$ in the outer expansion. The negative powers of y in the outer expansion, when transformed to inner variables, introduce terms of lower order in ϵ in the terms to be matched by the terms of the inner expansion. Each successive term of higher order in ϵ in the outer expansion contributes terms to the lowest and succeeding orders in the inner solution. In example problem 1, where $\sigma_1 = 1/\sigma_2 = \epsilon_c = \epsilon^{1/2}$, the first term of the inner expansion is

$$f^{1i} = \epsilon^{-1/2}[-Y + (Y^2 + 2)^{1/2}] = \epsilon^{-1/2}[Y^{-1} - (2Y^3)^{-1} + (2Y^5)^{-1} + \dots]$$

which can match to $(f^{10})^{1i} = \epsilon^{-1/2}Y^{-1}$ only to within algebraically decaying terms as $Y \rightarrow \infty$. Only additional higher orders in the outer expansion of f can account for these additional terms in the expansion of f^{1i} for large Y . The matching rule (77) allows for the extra algebraic terms.

A form of the matching principle, which is analogous to Kaplun's matching rule (45) in the conventional inner and outer expansion method, can be used in the present method in any of the classes of problems discussed above. If $(f^{ni})^{no}$ (defined after eq. (74)) is written in inner variables and then truncated at n terms, one has $[(f^{ni})^{no}]^{ni}$. Then a result corresponding to equation (45) is

$$[(f^{ni})^{no}]^{ni} = (f^{no})^{ni} \quad (78)$$

For problems in which $\delta \equiv 0$ (including all of class 1), equation (78) is equivalent to (45). As in equation (45), the displacements occurring in problems of class 5 are not retained in the matching by equation (78). The form (78) is conveniently used to evaluate constants of integration in a closed-form inner solution, but if explicit asymptotic outer boundary conditions for the inner problem must be known before the differential equations in the inner problem are solved, they are conveniently provided by the asymptotic form of the matching principle given by (76) or (77). It was noted above that if f^0 contains more than one displacement in class 5, then (76) does not apply. In that case, either (45) or (78) is suitable.

The important results of this section are summarized as follows:

$$\text{In classes 1, 2, 3: } f^{ni} = (f^0)^{ni} \quad (79a)$$

$$\text{In classes 4, 5: } f^{ni} \sim (f^0)^{ni} + \exp \quad \text{as } Y \rightarrow \infty \quad (79b)$$

$$\text{In classes 1, 2, 3, 4: } f^{ni} \sim (f^{no})^{ni} + \text{smaller alg} \quad \text{as } Y \rightarrow \infty \quad (80a)$$

$$\text{In class 5 with conditions (75): } f^{ni} \sim (f^{no})^{ni} + \exp \quad \text{as } Y \rightarrow \infty \quad (80b)$$

$$\text{In classes 1, 2, 3, 4, 5: } [(f^{ni})^{no}]^{ni} = (f^{no})^{ni} \quad (81)$$

Worthy of special note is the fact that, even for a problem in which it can be shown that a boundary-layer solution must go exponentially to an outer solution ((79b) is satisfied), the respective terms may approach algebraically

to the terms of the outer solution according to (80a) if the outer expansion contains negative powers of y . This occurs in class 4, of which two examples are provided above (p. 9).

The Composite Solution

The outer and inner solutions may be combined, if desired, into a single composite expansion that is uniformly valid to appropriate orders in both the outer and inner regions. A rule for forming the composite solution is directly analogous to equation (46). It may be stated as

$$f^{nc} = f^{ni} + f^{no} - (f^{no})^{ni} \quad (82)$$

where f^{nc} is the composite expansion to n th order. Note that f^{no} and $(f^{no})^{ni}$ in this method have a different meaning than in equation (46).

Matching of Several Functions

In a given problem one may wish (or need) to expand more than one function in asymptotic series and to match the several inner and outer functions. A singular perturbation problem may, of course, consist of a system of equations for which the order is reduced in the outer limiting process. Even in a problem with only one equation for one dependent variable f , the derivatives f_x, f_y, f_{yy} , etc., may be considered as different functions. Each such function, say g , should match according to the matching principle, with f, f^0, F , etc., replaced by g, g^0, G , etc., in the above development, and with σ_2 replaced by σ_g . When g is some function other than a derivative of f , σ_g can be determined from the governing equations.

Extension of the Method to a Class of Problems With Outer and Inner Functions Nonanalytic at $\epsilon = 0$

If it is suspected or found that nonpowers of ϵ_c will occur in some higher orders in the procedure outlined above, then the inner and outer functions can be redefined as the sums of convergent series like f^* in equation (49). For the purpose of matching by (76) in class 5, the class of problems considered should then be restricted to those for which the first N terms in the expansions, which are to be determined, contain only powers of ϵ_c . For the $(N + 1)$ st and higher order terms, (76) may not be applicable.

Consider a continuous function $f(x, y, \epsilon)$ with the asymptotic expansion (47), where $\varphi_1 = 1$ and $\varphi_n(\epsilon) = o[\varphi_{n-1}(\epsilon)]$ as $\epsilon \rightarrow 0$ and with the additional requirement on the asymptotic sequence $\{\varphi_n\}$ that for each $\varphi_n(\epsilon)$ there is some $m > 0$ such that

$$\epsilon^m = o[\varphi_n(\epsilon)] \quad \text{as } \epsilon \rightarrow 0 \quad (83)$$

Thus, the sequence $\{\varphi_n\}$ may contain terms with factors like $\epsilon^3(\log \epsilon)^{-1}$ or $\epsilon^6 \log \epsilon$, but excludes terms like, for example, $e^{-1/\epsilon}$. As in equation (49),

let f^* be the sum of the series of terms $\phi_n f_n$ only:

$$f^* = \sum_{n=1}^{\infty} \phi_n(\epsilon) f_n(x, y) \quad \text{in } D^*(x, y) \quad (84)$$

which converges for sufficiently small ϵ in some domain $D^*(x, y)$. (It can be shown (e.g., see ref. 32) that if (47) converges at some (x, y) , such as where a boundary condition is applied that either has a finite number of terms or is a convergent infinite-series expansion of the same form as in (47), then (47) converges in some neighborhood $D^*(x, y)$ of that location.) Then

$$f(x, y, \epsilon) = f^*(x, y, \epsilon) + q^*(x, y, \epsilon) \quad (85)$$

where

$$q^*(x, y, \epsilon) \sim 0 + \exp \quad \text{as } \epsilon \rightarrow 0 \quad (86)$$

A convenient illustration of a typical function f^* can be made as in equations (51) to (53) by adding a term $\epsilon^4 \log \epsilon / (1 + y)$ to the right side of equations (51a) and (51c). In terms of $y \equiv \epsilon Y$, equations (52a) and (53b) would then have an additional term $\epsilon^4 \log \epsilon / (1 + \epsilon Y)$ on the right side and equation (52b) would have an additional $\sum_{j=0}^{\infty} (-1)^j Y^j (\epsilon^{j+4} \log \epsilon)$ on the right.

For the asymptotic expansion method, outer and magnified inner functions that are sums of convergent series in some domain are used (like f^* and F in equations (51) to (53) with the additions noted above). Thus, it is assumed that

$$f(x, y, \epsilon) = f^o(x, y, \epsilon) + q^o(x, y, \epsilon)$$

where

$$f^o(x, y, \epsilon) = \sum_{n=1}^{\infty} \phi_n(\epsilon) f_n(x, y) \quad \text{in some } D^o(x, y) \quad (87)$$

$$\phi_1 = 1, \phi_n(\epsilon) = o[\phi_{n-1}(\epsilon)], \epsilon^m = o[\phi_n(\epsilon)], \quad m > 0 \quad \text{as } \epsilon \rightarrow 0;$$

$$q^o(x, y, \epsilon) \sim 0 + \exp \quad \text{as } \epsilon \rightarrow 0 \quad \text{with } x, y \text{ fixed}$$

and that

$$\left. \begin{aligned}
f(x, Y, \epsilon) &= f^i(x, Y, \epsilon) + q^i(x, Y, \epsilon) \\
\text{where} \\
Y &= y/\sigma_1(\epsilon) \\
F(x, Y, \epsilon) &= f^i(x, Y, \epsilon)/\sigma_2(\epsilon) \\
&= \sum_{n=1}^{\infty} \Phi_n(\epsilon) F_n(x, Y) \quad \text{in some } D^i(x, Y) \\
\Phi_1 &= 1, \Phi_n(\epsilon) = o[\Phi_{n-1}(\epsilon)], \epsilon^m = o[\Phi_n(\epsilon)], \quad m > 0 \quad \text{as } \epsilon \rightarrow 0; \\
q^i(x, Y, \epsilon) &\sim 0 + \exp \quad \text{as } \epsilon \rightarrow 0 \quad \text{with } x, Y \text{ fixed}
\end{aligned} \right\} \quad (88)$$

The remainder of the development is the same as in the preceding subsections (pp. 16 ff) except that the asymptotic sequences $\{\varphi_n(\epsilon)\}$ and $\{\Phi_n(\epsilon)\}$ must be determined by some means as in the conventional inner- and outer-expansion approach (ref. 1). Since the satisfaction by f^0 of a boundary condition on f determines the displacement $\delta(x, \epsilon)$ (see subsections entitled "Definitions of inner and outer functions," pp. 21-25; and "Displacement Variables," pp. 27-29), the displacement may have the expansion (cf. eq. (64)).

$$\delta \equiv \sigma_\delta(\epsilon) \Delta(x, \epsilon) = \sigma_\delta(\epsilon) \sum_{n=1}^{\infty} \varphi_n(\epsilon) \Delta_n(x) \quad (89)$$

The displacement variables are then

$$\left. \begin{aligned}
\tilde{y} &= y - \delta = \sigma_1 Y - \sigma_\delta \sum_{n=1}^{\infty} \varphi_n(\epsilon) \Delta_n(x) \\
\tilde{x} &= x
\end{aligned} \right\} \quad (90a)$$

$$\tilde{f}^0(\tilde{x}, \tilde{y}, \epsilon) = f^0(x, y, \epsilon) \quad (90b)$$

and the expansion of \tilde{f}^0 is

$$\tilde{f}^0(\tilde{x}, \tilde{y}, \epsilon) = \sum_{n=1}^{\infty} \varphi_n(\epsilon) \tilde{f}_n(\tilde{x}, \tilde{y}) \quad (91)$$

From equations (87) and (88), equation (70) still applies and $(q^0)^i - q^i$ contains only terms that are changed from order exponentially small as $\epsilon \rightarrow 0$ with x, y fixed to the order of some $\Phi_n(\epsilon)$ as $\epsilon \rightarrow 0$ with x, Y fixed. These terms then decay exponentially to zero as $Y \rightarrow \infty$. Thus equation (71) still applies, and to order $\sigma_2(\epsilon)\Phi_n(\epsilon)$ in inner variables, equation (72) applies. Now let f^{no} denote n terms of $\tilde{f}^0(\tilde{x}, \tilde{y}, \epsilon)$:

$$f^{no} = \sum_{m=1}^n \Phi_m(\epsilon) \tilde{f}_m(\tilde{x}, \tilde{y}) \quad (92a)$$

and

$$f^{ni} = \sigma_2(\epsilon) \sum_{m=1}^n \Phi_m(\epsilon) F_m(x, Y) \quad (92b)$$

and let $(f^{no})^{ni}$ denote the terms to order $\sigma_2(\epsilon)\Phi_n(\epsilon)$ in equation (92a) after it has been transformed by equation (90a) to x, Y variables. Then the matching rule (76) applies if f^{no} contains all the parts of f^0 that would appear in $(f^0)^{ni}$; that is, f^0 can be replaced by f^{no} in equation (72) if this condition is satisfied. At least to some order N for which

$$\varphi_n(\epsilon) = \Phi_n(\epsilon) = [\epsilon_c(\epsilon)]^{n-1} \quad \text{for all } n \leq N \quad (93)$$

this condition is satisfied if conditions (75) are met. If equation (93) is true for some N but conditions (75) are not met, then either an intermediate expansion is needed to replace f^{no} in equation (76) or else equation (77) applies (for $n \leq N$), as discussed under The Matching Principle.

If for some N it is found that for $n > N$ terms with nonpowers of ϵ_c occur, then at least Kaplun's standard matching rule (eq. (45) or (78)) can be used.

APPLICATION OF THE METHOD WITH DISPLACEMENT VARIABLES TO HYPERSONIC VISCOUS FLOW OVER A SPHERE

The purpose of this section is to apply the method developed above to the problem of hypersonic viscous flow over a sphere at high Reynolds number. This application is limited to cases for which external (shock-generated) vorticity has only second-order (and higher) effects on the solution for the viscous flow near the body. It will be seen that this limitation requires that the shock density ratio be not too small. (A list of references on the viscous blunt-body problem is given in reference 30.)

It has been pointed out (ref. 30) that the second-order effects of external vorticity and displacement are "kindred effects" and that they are "global in nature" and, as such, are much more difficult to calculate than other second-order effects. In the point of view of the present approach, the presence of external vorticity is responsible for higher order displacements,

so that the inner boundary for the outer problem is affected to each successive order of the small parameter. Using the present approach, the displacement effects are determined in the matching process and subsequent solution of the inner problems.

After definition of the mathematical model for the problem to be solved, and subsequent application of the procedure to obtain the asymptotic expansions, the results will be compared with a numerical solution of the model problem.

The Mathematical Model for Hypersonic Viscous Flow Over a Sphere

This approximate application to viscous flow is not intended as a thorough treatment of the hypersonic blunt-body problem for all regimes of flight (see, e.g., analyses in refs. 27 and 36-46). The object here is to isolate the role of the matching principle as much as possible and, in particular, the role of displacement in the matching. The mathematical model will therefore be kept as simple as possible so that the basic principles will not be obscured by too much detail. To achieve this, a number of important aspects of some regimes of flight in the physical problem are ignored in favor of the desired simplicity. The model must, however, retain the essential character that we are studying. Simplifications that do not change the essential character of the problem are the assumptions of: (1) constant density and viscosity behind the shock wave, (2) local similarity near the stagnation streamline (includes shock wave concentric with the body), and (3) discontinuous properties across the shock wave, determined by Rankine-Hugoniot equations. Also, the low-density effects of velocity slip and temperature jump are specifically neglected (ref. 47). All these assumptions and approximations have been used and discussed by various authors. Included in the characteristics of the problem that must be retained are: curved body, curved detached shock wave, and consideration of the full governing equations (Navier-Stokes). In the present treatment, curvature effects will be retained implicitly (by a transformation of coordinates), and need not be considered as second order or otherwise. The gas immediately behind the shock wave can be assumed to be in thermodynamic equilibrium for evaluation of conditions there. The flow is assumed to be steady.

The Navier-Stokes equations for conservation of mass and momentum are:

$$\nabla \cdot (\rho \vec{V}) = 0 \quad (94)$$

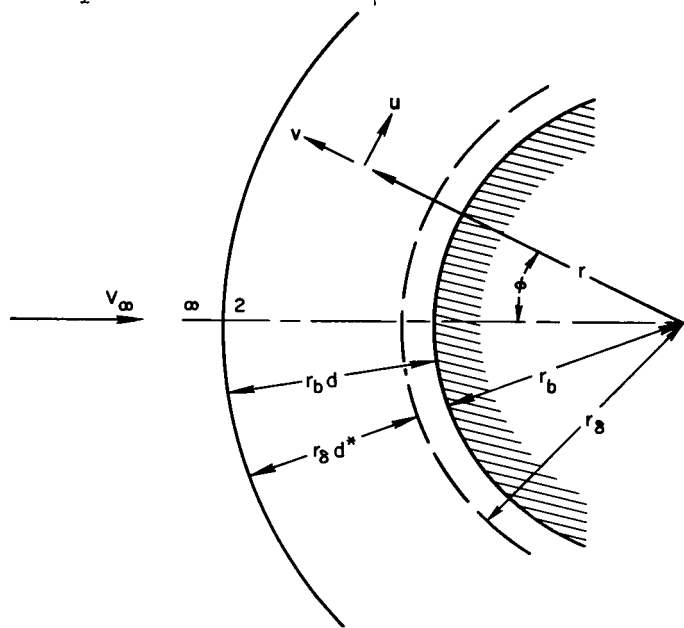
$$\rho(\vec{V} \cdot \nabla)\vec{V} + \nabla p = \mu \nabla^2 \vec{V} \quad (95)$$

where ∇ is the vector operator "grad," ρ the local mass density, p the pressure, \vec{V} the velocity vector, and μ the viscosity coefficient (assumed constant), and where the equation of state for the flow between the shock wave and the body is taken to be

$$\rho = \text{constant} = \rho_2 \quad (96)$$

(subscript 2 denotes a value immediately behind the shock on the axis of symmetry). Because of equation (96), the energy equation is uncoupled from the others and is not needed in the present analysis (except in finding conditions across the shock wave).

Consider axisymmetric spherical coordinates (r, ϕ) with r measured from the sphere center and ϕ measured from the forward axial streamline (fig. 3).



Let the radius of the spherical body be r_b and that of the detached concentric shock wave (near the axis) be $r_s = r_b(1 + d)$. Denote free-stream conditions by subscript ∞ . Let the velocity components, respectively tangential and normal to the body, be $V_\infty u$ and $V_\infty v$, so that u and v are dimensionless and $V \equiv V_\infty \sqrt{u^2 + v^2}$ is the velocity magnitude. Define a dimensionless pressure function $\bar{p} \equiv (p - p_\infty) / \rho_\infty V_\infty^2$; and denote the shock density ratio by $k = \rho_\infty / \rho_2$. The boundary conditions for the flow near the axis (taken to be the no-slip surface conditions and the Rankine-Hugoniot shock conditions) in terms of these dimensionless variables are:

Figure 3.- Configuration for sphere in hypersonic flow.

At $r = r_b$:

$$u = v = 0 \quad (97)$$

At $r = r_s = r_b(1 + d)$:

$$v = -k \cos \phi \quad (98a)$$

$$u = \sin \phi \quad (98b)$$

$$\bar{p} = (1 - k) \cos^2 \phi \quad (98c)$$

For complete determination of conditions behind the shock, one needs to include also in the Rankine-Hugoniot conditions the energy relation across the shock:

At $r = r_s$:

$$h + \frac{1}{2} V^2 = h_\infty + \frac{1}{2} V_\infty^2 \quad (98d)$$

where h is the specific enthalpy, and an equation of state

$$h = h(p, \rho) \quad (98e)$$

that may be represented, for example, by equilibrium air charts or tables.

If equations (94) and (95) are put in terms of the coordinates r and φ , conservation of mass (eq. (94)) is satisfied by definition of a dimensionless Stokes stream function ψ by

$$\left. \begin{aligned} r \frac{\partial \psi}{\partial r} &= u \left(\frac{r}{r_b} \right)^2 \sin \varphi \\ \frac{\partial \psi}{\partial \varphi} &= -v \left(\frac{r}{r_b} \right)^2 \sin \varphi \end{aligned} \right\} \quad (99)$$

For convenience, we define a transformation

$$\zeta \equiv \log(r/r_b) \quad (100a)$$

so that

$$r \frac{\partial(\quad)}{\partial r} = \frac{\partial(\quad)}{\partial \zeta} \quad (100b)$$

(Note that in terms of physical distance from the body surface, $r_b y$, the new variable ζ is

$$\zeta \equiv \log(1 + y) = y + O(y^2) \quad (100c)$$

and so is the same as y to first order for small distances. Thus, the variable ζ is convenient both as a spherical coordinate in the outer problem, since it will simplify the equations, and as a boundary-layer variable.) With a Reynolds number defined as

$$R \equiv \frac{\rho_\infty V_\infty r_b}{\mu_2} = \frac{1}{k} \frac{\mu_\infty}{\mu_2} \left(\frac{\rho_\infty V_\infty r_b}{\mu_\infty} \right) \quad (101)$$

the components of the momentum equation (95) become

$$\begin{aligned} uu_\varphi + vu_\zeta + uv - R^{-1}e^{-\zeta}[u_\zeta\zeta + u_\zeta + 2v_\varphi - u(\sin \varphi)^{-2} \\ + (\cot \varphi)u_\varphi + u_{\varphi\varphi}] = -k\bar{p}_\varphi \end{aligned} \quad (102a)$$

$$\begin{aligned} uv_\varphi + vv_\zeta - u^2 - R^{-1}e^{-\zeta}[v_\zeta\zeta + v_\zeta - 2v + v_{\varphi\varphi} - 2u_\varphi \\ + (\cot \varphi)(v_\varphi - 2u)] = -k\bar{p}_\zeta \end{aligned} \quad (102b)$$

where subscripts φ and ζ denote partial differentiation with respect to the corresponding independent variables. Equations (102) may then be combined by cross-differentiating and by equating $\bar{p}_{\zeta\varphi}$ to $\bar{p}_{\varphi\zeta}$ to eliminate pressure.

Equations (99) with the transformation (100) may then be inserted to obtain a nonlinear partial differential equation for the single dependent variable ψ in terms of the independent variables φ, ζ and the parameter R . The result is

$$\begin{aligned}
& \psi_\xi \psi_{\xi\xi\phi} - 2(\cot \phi) \psi_\xi \psi_{\xi\xi} - \psi_\xi \psi_{\xi\phi} + 2(\cot \phi) \psi_\xi^2 + \psi_\xi \psi_{\phi\phi\phi} \\
& - 3(\cot \phi) \psi_\xi \psi_{\phi\phi} + 3(\cot^2 \phi - 1) \psi_\phi \psi_\xi - \psi_\phi \psi_{\xi\xi\xi} - \psi_\phi \psi_{\phi\phi\xi} \\
& + 5\psi_\phi \psi_{\xi\xi} + 4\psi_\phi \psi_{\phi\phi} + (\cot \phi) \psi_\phi \psi_{\phi\xi} - 4(\cot \phi) \psi_\phi^2 \\
& = R^{-1} e^\xi \sin \phi [\psi_{\xi\xi\xi\xi} - 6\psi_{\xi\xi\xi} + 11\psi_{\xi\xi} - 6\psi_\xi - 2(\cot \phi) \psi_{\xi\xi\phi} \\
& + 2\psi_{\xi\xi\phi\phi} - 6\psi_{\xi\phi\phi} + 6(\cot \phi) \psi_{\xi\phi} - (9 \cot \phi + 3 \cot^3 \phi) \psi_\phi \\
& + (8 + 3 \cot^2 \phi) \psi_{\phi\phi} - 2(\cot \phi) \psi_{\phi\phi\phi} + \psi_{\phi\phi\phi\phi}] \quad (103)
\end{aligned}$$

The boundary conditions for ψ , obtained from equations (97) and (98), are:

At $\xi = 0$:

$$\psi_\xi = \psi_\phi = 0 \quad (104a)$$

At $\xi = \xi_s = \log(1 + d)$:

$$\psi_\phi = k(1 + d)^2 \sin^2 \phi \cot \phi \quad (105a)$$

$$\psi_\xi = (1 + d)^2 \sin^2 \phi \quad (105b)$$

and

$$\begin{aligned}
& \psi_\xi \psi_{\xi\phi} - (\cot \phi) \psi_\xi^2 - \psi_\phi \psi_{\xi\xi} + \psi_\phi \psi_\xi - \nu R^{-1} (1 + d) \sin \phi [\psi_{\xi\xi\xi} \\
& - 3\psi_{\xi\xi} + 2\psi_\xi + \psi_{\xi\phi\phi} - (\cot \phi) \psi_{\xi\phi} - 2\psi_{\phi\phi} + 2(\cot \phi) \psi_\phi] \\
& = 2k(1 - k)(1 + d)^4 \sin^4 \phi \cot \phi \quad (105c)
\end{aligned}$$

where one can take either $\nu = 0$ or $\nu = 1$ for the model (discussed below). Without loss of generality one can, of course, replace the condition $\psi_\phi = 0$ in conditions (104a) with

$$\xi = 0, \quad \psi = 0 \quad (104b)$$

Condition (105c) with $\nu = 1$ was obtained by first differentiating condition (98c) with respect to ϕ and combining the result with equation (102a). Thus, $\nu = 1$ implies strict application of the Rankine-Hugoniot shock conditions (eq. (98)). However, since those conditions are derived assuming inviscid flow on both sides of the shock, one might just as well take $\nu = 0$ (ref. 48). The condition with $\nu = 1$ is more consistent with the Rankine-Hugoniot conditions, but the condition with $\nu = 0$ is simpler. In any case, the value of ν does not affect the perturbation problem in the first- and second-order solutions (refs. 46 and 48). (Some subsequent work

shows that this is true only if, as assumed here, the vorticity interaction is not greater than a second-order effect.) Moreover, the differences in results calculated for the two values of ν in the exact solution could be taken as an indicator of the influence of neglecting the shock-wave-viscosity effects on the flow behind the shock.

Equation (103) and boundary conditions (104) and (105) contain two independent parameters, k and R . The dimensionless shock detachment distance, d , is an unknown function of k and R , to be determined in the solution.

Solution of the Problem by Assumed Power-Series Expansions With Displacement Variables

The problem of equations (103), (104), and (105) is to be solved asymptotically for $\psi(\varphi, \xi, k, l/R)$ as $l/R \rightarrow 0$. The parameter k is assumed to be nonvanishing as $l/R \rightarrow 0$. The result of substituting

$$\psi(\varphi, \xi, k, l/R) \sim \psi_1(\varphi, \xi, k) \quad \text{as } l/R \rightarrow 0 \quad (106)$$

into equations (103), (104), and (105) is that the order of the partial differential equation is reduced and the no-slip condition, $\psi_\xi = 0$ on $\xi = 0$, cannot be satisfied. This problem is known (and will be seen) to belong to class 5, as defined above (pp. 5-9), and will be treated as a problem of that class.

The outer function representing the stream function. - According to the method outlined above (cf. eq. (55a)), let

$$\psi(\varphi, \xi, k, l/R) = \psi^0(\varphi, \xi, k, l/R) + q^0(\varphi, \xi, k, l/R) \quad (107)$$

and assume

$$d(k, l/R) = d^0(k, l/R) + q_d(k, l/R) \quad (108)$$

where ψ^0 and d^0 are assumed to be analytic with respect to some $\epsilon = \epsilon(l/R)$ at $l/R = 0$ so that

$$\psi^0(\varphi, \xi, k, l/R) = \psi_1(\varphi, \xi, k) + \epsilon \psi_2(\varphi, \xi, k) + \epsilon^2 \psi_3(\varphi, \xi, k) + \dots \quad (109)$$

$$d^0(k, l/R) = d_1(k) + \epsilon d_2(k) + \epsilon^2 d_3(k) + \dots \quad (110)$$

and where

$$\left. \begin{aligned} q^0 &\sim 0 + \exp && \text{as } l/R \rightarrow 0 \text{ with } \varphi, \xi, k \text{ fixed} \\ q_d &\sim 0 + \exp && \text{as } l/R \rightarrow 0 \text{ with } k \text{ fixed} \end{aligned} \right\} \quad (111)$$

Since the limit function ψ_1 in (106) was found to be a nonuniform representation of ψ at $\xi = 0$, by equation (107), the outer function ψ^0 represents

ψ to within exponentially small amounts (q^0) as $\epsilon \rightarrow 0$ everywhere except at $\zeta = 0$. Then substitution of equations (107) and (108) into (103), (104), and (105) shows that ψ^0 must satisfy equations (103), (104), and (105) with d replaced by d^0 and with ψ replaced by ψ^0 at all ζ except very near $\zeta = 0$. According to the method outlined above, then, since $w_1 = \psi_1 \zeta$ is the function that cannot satisfy a boundary condition at $\zeta = 0$, that condition is not required of $w^0 \equiv \psi_\zeta^0$; but the condition $\psi = 0$ can also be required of ψ^0 if it is allowed to be satisfied at a displaced boundary $\zeta = \delta$ (see p. 23), with the displacement δ to be determined by compatibility with the inner solution. Assuming that δ is real in this problem (see subsection "Displacement Variables," pp. 27-29), we can write

$$\delta/\sigma_1 = \Delta(\varphi, k, l/R) = \Delta_1(\varphi, k) + \epsilon \Delta_2(\varphi, k) + \epsilon^2 \Delta_3(\varphi, k) + \dots \quad (112)$$

where $\sigma_1 = \sigma_1(l/R)$ (to be determined) represents the order of magnitude of the width of the nonuniform region measured in units of ζ . The problem for ψ^0 has now been derived as: equation (103) with ψ replaced by ψ^0 ; and the boundary conditions:

$$\zeta = \sigma_1 \Delta, \quad \psi^0 = 0 \quad (113a)$$

$$\zeta = \log(1 + d^0):$$

$$\psi_\varphi^0 = k(1 + d^0)^2 \sin^2 \varphi \cot \varphi \quad (113b)$$

$$\psi_\zeta^0 = (1 + d^0)^2 \sin^2 \varphi \quad (113c)$$

and

$$\{\text{condition (105c) with } \psi \text{ replaced by } \psi^0 \text{ and } d \text{ by } d^0\} \quad (113d)$$

The quantities $\epsilon = \epsilon(l/R)$, $\sigma_1 = \sigma_1(l/R)$, $\Delta = \Delta(\varphi, k, l/R)$, and $d^0(k, l/R)$ are to be determined. Note that $\sigma_1 \Delta$ is a displacement of the independent variable ζ and is therefore proportional to the actual displacement distance to first order only (see eq. (100c)).

A dimensionless magnified viscous-flow displacement, Δ' , of the y coordinate may be defined, with use of equation (100c), by

$$\Delta' \equiv y_\delta/\sigma_1 \equiv (e^\delta - 1)/\sigma_1 \quad (114)$$

The displacement distance is related to the shock standoff distance as follows. Denote by $r_\delta \equiv r_b(y_\delta + 1)$ the radius of the frictionless body that would produce the flow field that corresponds to the actual outer flow when viscosity is present (cf. ref. 33). The shock standoff distance from the equivalent frictionless body is denoted by $r_\delta d^*$ (fig. 3); from the actual body, it is $r_b d$. (For $\nu = 0$, $d^* = d_1$ (given by the inviscid solution of Lighthill)). For $\nu = 1$, d^* will depend on R , but not to the order to be considered here, that is, $d^* = d_1 + \nu O(1/R)$. However, the complete solution for d^* is shown at the end of this section for the comparison of the asymptotic expansion for Δ' with the exact solution of the model problem.) The shock radius is then

$$r_s = r_b(1 + d) = r_\delta(1 + d^*) \quad (115)$$

The dimensionless magnified displacement is then given by

$$\Delta' \equiv \frac{y_\delta}{\sigma_1} = \frac{r_\delta - r_b}{\sigma_1 r_b} = \frac{d - d^*}{\sigma_1(1 + d^*)} \quad (116)$$

Equating (114) and (116) gives the relationship between $d - d^*$ and δ :

$$d - d^* = (1 + d^*)(e^\delta - 1) \quad (117)$$

Note now from equation (117) that, since d^* and δ are analytic with respect to ϵ_c at $\epsilon = 0$, so also is d . Then $q_d = 0$ in equation (108) and

$$d^0 = d = d_1(k) + \epsilon d_2(k) + \epsilon^2 d_3(k) + \dots \quad (118)$$

The assumptions of the concentric shock and local similarity of the flow solution require the dimensionless standoff distance, d , and r_δ both to be functions only of k and R , not varying with ϕ .

The displacement variables $\tilde{\phi}$ and $\tilde{\zeta}$ are defined (cf. eqs. (64) and (67)) by:

$$\tilde{\phi} \equiv \phi, \quad \tilde{\zeta} \equiv \zeta - \sigma_1 \Delta \quad (119a)$$

The outer displacement dependent variable (cf. eqs. (68) and (69)) is defined by

$$\tilde{\psi}^0(\tilde{\phi}, \tilde{\zeta}, k, l/R) \equiv \psi^0(\phi, \zeta, k, l/R) \quad (119b)$$

$$= \tilde{\psi}_1(\tilde{\phi}, \tilde{\zeta}, k) + \epsilon \tilde{\psi}_2(\tilde{\phi}, \tilde{\zeta}, k) + \epsilon^2 \tilde{\psi}_3(\tilde{\phi}, \tilde{\zeta}, k) + \dots \quad (119c)$$

Substitution of equations (109), (112), and (118) into the differential equation for ψ^0 , (103), and into the conditions (113) and letting $l/R \rightarrow 0$ (so that also the as yet unknown quantities ϵ and σ_1 vanish) gives the problem for ψ_1 . In finding the boundary conditions for ψ_1 it is assumed that ψ^0 is analytic with respect to ζ at $\zeta = \sigma_1 \Delta$ and at ζ_s . The condition at $\zeta = \sigma_1 \Delta$ is then transferred to $\zeta = 0$ by Maclaurin's series, from which

$$\psi^0(\phi, \sigma_1 \Delta) = \psi^0(\phi, 0) + \sigma_1 \Delta \psi_\zeta^0(\phi, 0) + (1/2!)(\sigma_1 \Delta)^2 \psi_{\zeta\zeta}^0(\phi, 0) + \dots = 0 \quad (120)$$

and into which equations (109) and (112) must be substituted. Similarly, the conditions at the shock [$\zeta = \zeta_s \equiv \log(1 + d)$] are expressed in terms of conditions at

$$\zeta_{s1} \equiv \log(1 + d_1) \quad (121a)$$

where

$$\zeta_s - \zeta_{s1} = \epsilon \left(\frac{d_2}{1 + d_1} \right) + \epsilon^2 \left[\frac{d_3}{1 + d_1} - \frac{1}{2} \left(\frac{d_2}{1 + d_1} \right)^2 \right] + \dots \quad (121b)$$

by means of Taylor's series.

Solution of the first outer problem. - The resulting problem for ψ_1 is

$$\begin{aligned} & \psi_{1\zeta} \psi_{1\zeta\zeta\varphi} - 2(\cot \varphi) \psi_{1\zeta} \psi_{1\zeta\zeta} - \psi_{1\zeta} \psi_{1\zeta\varphi} + 2(\cot \varphi) \psi_{1\zeta}^2 + \psi_{1\zeta} \psi_{1\varphi\varphi\varphi} \\ & - 3(\cot \varphi) \psi_{1\zeta} \psi_{1\varphi\varphi} + 3(\cot^2 \varphi - 1) \psi_{1\varphi} \psi_{1\zeta} - \psi_{1\varphi} \psi_{1\zeta\zeta\zeta} - \psi_{1\varphi\varphi\zeta} \psi_{1\varphi} \\ & + 5\psi_{1\varphi} \psi_{1\zeta\zeta} + 4\psi_{1\varphi} \psi_{1\varphi\varphi} + (\cot \varphi) \psi_{1\varphi} \psi_{1\varphi\zeta} - 4(\cot \varphi) \psi_{1\varphi}^2 = 0 \end{aligned} \quad (122a)$$

with the conditions

$$\psi_1(\varphi, 0) = 0 \quad (122b)$$

$$\psi_{1\zeta}(\varphi, \zeta_{s1}) = (1 + d_1)^2 \sin^2 \varphi \quad (122c)$$

$$\psi_{1\varphi}(\varphi, \zeta_{s1}) = k(1 + d_1)^2 \sin^2 \varphi \cot \varphi \quad (122d)$$

and

$$\begin{aligned} & \left[\psi_{1\zeta} \psi_{1\varphi\zeta} - (\cot \varphi) \psi_{1\zeta}^2 - \psi_{1\varphi} \psi_{1\zeta\zeta} + \psi_{1\varphi} \psi_{1\zeta} \right]_{\zeta=\zeta_{s1}} \\ & = 2k(1 - k)(1 + d_1)^4 \sin^4 \varphi \cot \varphi \end{aligned} \quad (122e)$$

This problem has the separable solution

$$\psi_1 = (\sin^2 \varphi) f_1(\zeta) \quad (123)$$

where $f_1(\zeta)$ is found from the separated ordinary differential equation

$$f_1''' - 5f_1'' + 2f_1' + 8f_1 = 0 \quad (124)$$

(where $()'$ generally denotes differentiation of a function with respect to its argument) and has the form

$$f = C_{11}e^{4\zeta} + C_{12}e^{2\zeta} + C_{13}e^{-\zeta} \quad (125a)$$

$$= C_{11}(r/r_b)^4 + C_{12}(r/r_b)^2 + C_{13}(r/r_b)^{-1} \quad (125b)$$

$$= a_{10} + a_{11}\zeta + a_{12}\zeta^2 + a_{13}\zeta^3 + \dots \quad (125c)$$

This is the well-known solution of Lighthill (ref. 49) for the inviscid flow over a sphere. The boundary conditions determine that

$$\left. \begin{aligned} C_{11} &= (1 - k)^2/10k(1 + d_1)^2 \\ C_{12} &= (4k - 1)/6k \\ C_{13} &= (1 - k)(1 - 6k)(1 + d_1)^3/15k \end{aligned} \right\} \quad (126a)$$

and that the function $d_1 = d_1(k)$ is the solution of the algebraic equation obtained from

$$C_{11} + C_{12} + C_{13} = 0 \quad (126b)$$

The constants in equation (125c) will be needed for matching to the inner solution and are easily found to be:

$$a_{10} = 0 \quad (127a)$$

$$a_{11} = 4C_{11} + 2C_{12} - C_{13} = \frac{1}{2k} \left[\frac{(1 - k)^2}{(1 + d_1)^2} + 4k - 1 \right] \quad (127b)$$

$$a_{12} = 8C_{11} + 2C_{12} + \frac{1}{2} C_{13} = \frac{1}{4k} \left[\frac{3(1 - k)^2}{(1 + d_1)^2} + 4k - 1 \right] \quad (127c)$$

$$\begin{aligned} a_{13} &= \frac{32}{3} C_{11} + \frac{4}{3} C_{12} - \frac{1}{6} C_{13} = \frac{1}{12k} \left[\frac{13(1 - k)^2}{(1 + d_1)^2} + 3(4k - 1) \right] \\ &= -\frac{a_{11}}{3} + 5\frac{a_{12}}{3} \end{aligned} \quad (127d)$$

The significance of the constants a_{11} and a_{12} can be observed from the fact that the dimensionless velocity u and the dimensionless vorticity $\omega \equiv (r_b/V_\infty)(\nabla \times \vec{V})$ at the body surface in the first-order inviscid solution are:

$$u_{1b} = a_{11} \sin \varphi, \quad \omega_{1b} = (a_{11} - 2a_{12}) \sin \varphi \quad (128)$$

To find the terms for the higher order outer solutions (ψ_2, ψ_3 , etc.) the parameters $\epsilon = \epsilon(1/R)$ and $\sigma_1 = \sigma_1(1/R)$ must be known. Therefore, the inner problem (of determining the inner function ψ^i , to be defined) must be formulated. Outer boundary conditions for the inner problem will need to be supplied by the matching principle which, in the present approach, uses the displacement variables. For that purpose, the first term in the outer displacement expansion (119c) is found as follows: From equations (109), (119b), and (119c),

$$\tilde{\psi}_1(\tilde{\varphi}, \tilde{\zeta}, k) = \psi_1(\varphi, \zeta, k) + O(\epsilon) \quad (129)$$

Into the right side of equation (129) we then substitute (123), (125c), and the transformation (119a) to obtain

$$\tilde{\psi}_1 = \sin^2 \tilde{\varphi} (a_{11}\tilde{\zeta} + a_{12}\tilde{\zeta}^2 + a_{13}\tilde{\zeta}^3 + \dots) \quad (130)$$

The inner function representing the stream function.- The inner function ψ^i , which represents ψ , at least asymptotically, in the inner region as $1/R \rightarrow 0$, is found by magnifying the variables in such a way that the nonuniformity of the problem (reduced-order differential equation with loss of a boundary condition as $1/R \rightarrow 0$) is removed. Following the method outlined above, one can define magnified variables Z and Ψ such that (cf. eqs. (60))

$$Z = \zeta/\sigma_1 \quad (131a)$$

and

$$\Psi(\varphi, Z, k, 1/R) = (1/\sigma_2)\psi^i(\varphi, Z, k, 1/R) \quad (131b)$$

where

$$\psi(\varphi, \zeta, k, 1/R) = \psi^i(\varphi, Z, k, 1/R) + q^i(\varphi, Z, k, 1/R) \quad (132)$$

and Ψ is assumed to be analytic with respect to some $\epsilon = \epsilon(1/R)$ at $1/R = 0$ so that

$$\Psi(\varphi, Z, k, 1/R) = \Psi_1(\varphi, Z, k) + \epsilon\Psi_2(\varphi, Z, k) + \epsilon^2\Psi_3(\varphi, Z, k) + \dots \quad (133)$$

and where

$$q^i(\varphi, Z, k, 1/R) \sim 0 + \exp \quad \text{as } 1/R \rightarrow 0 \text{ with } \varphi, Z, k \text{ fixed} \quad (134)$$

The quantities $\epsilon = \epsilon(1/R)$, $\sigma_1 = \sigma_1(1/R)$, and $\sigma_2 = \sigma_2(1/R)$ are to be determined.

Substitution of equations (131) through (134) into the boundary conditions (104a) and (104b) shows that Ψ must satisfy the conditions

$$\Psi = \Psi_Z = 0 \quad \text{at } Z = 0 \quad (135)$$

To determine σ_1 and σ_2 , consider the rules stated under Magnifying Factors. First, since the function that does not satisfy one of the inner

boundary conditions in the outer solution is $w^0 = \psi_\zeta^0$, we require that ψ_ζ^1 be $O(1)$ in the inner region (as $1/R \rightarrow 0$ with Z fixed). Then, from equations (131)

$$\psi_\zeta^1 = (\sigma_2/\sigma_1)\Psi_Z = O(1) \quad \text{as } 1/R \rightarrow 0 \text{ with } Z \text{ fixed}$$

and, since $\Psi_Z = O(1)$ as $1/R \rightarrow 0$ in the inner region, we take, without loss of generality,

$$\sigma_1 = \sigma_2 = \sigma \quad (136a)$$

If equations (131) through (134) are substituted into the differential equation (103), and if the equation is then multiplied by σ , and $1/R$ is made to go to zero, application of the principle of least degeneracy requires that

$$\sigma^2 R = O(1) \quad \text{as } 1/R \rightarrow 0$$

so that

$$\sigma = R^{-1/2} \quad (136b)$$

The inner problem is the problem for $\Psi(\phi, Z, k, 1/R)$ which is composed of conditions (135), additional outer conditions to be supplied by the matching principle, and the differential equation (103) with equations (131) and (132) substituted. Except for the outer boundary conditions, the inner problem for Ψ contains only integral powers of $1/R$. However, the outer problem contains $\sigma = R^{-1/2}$ in its inner boundary condition, and it will be seen that $R^{-1/2}$ also occurs explicitly in the outer conditions supplied to the inner problem by the matching principle. Hence, as described above, we then assume that

$$\epsilon = R^{-1/2} \quad (137)$$

is the form of the small parameter with respect to which ψ^0 and Ψ are analytic at $\epsilon = 0$.

Solution of the second outer problem. - The problems for the respective terms of the outer expansion (109) are determined by substituting equations (109), (110), and (118) into the problem for ψ^0 as stated in the subsection "The outer function representing the stream function," pp. 41-44. The problem for ψ_1 is given by equations (122) with the solution given by equations (123) through (126). The problem for ψ_2 contains a great deal more terms than the problem for ψ_1 , and so will not be written out here. Note, however, that the condition on ψ_2 from equation (113a) is

$$\psi_2(\phi, 0) = -\Delta_1 \psi_{1\zeta}(\phi, 0)$$

which contains the unknown Δ_1 . The problem for ψ_2 can be solved at this point if the form of Δ_1 is known. To continue the assumption of local similarity in the higher-order outer solutions, we need to have the displacement surface concentric with the body near the axis (i.e., $\Delta = \text{constant}$), like the shock wave. Let us carry on with this assumption to find the second

outer solution, but then solve the first inner problem, without making the assumption a priori there, to determine the actual significance of assuming a concentric displacement surface (i.e., of taking Δ_1 to be independent of ϕ near $\phi = 0$). The second outer problem has the separable solution (found in a different form by Van Dyke, ref. 30):

$$\psi_2 = (\sin^2 \phi) f_2(\xi) \quad (138)$$

where $f_2(\xi)$ is the solution to the ordinary differential equation

$$f_2''' - 5f_2'' + 2f_2' + 8f_2 = 0 \quad (139)$$

Equation (139) has the same form as (124), and so has the solution

$$f_2(\xi) = C_{21}e^{4\xi} + C_{22}e^{2\xi} + C_{23}e^{-\xi} \quad (140a)$$

$$= a_{20} + a_{21}\xi + a_{22}\xi^2 + \dots \quad (140b)$$

The boundary conditions for ψ_2 , and those for ψ_1 , lead to determination of the constants C_{21} , C_{22} , C_{23} , and d_2 . They are found to be

$$\left. \begin{aligned} C_{21} &= -(1-k)^2 d_2 / 5k(1+d_1)^3, & C_{22} &= 0 \\ C_{23} &= (1-k)(1-6k)d_2(1+d_1)^2 / 5k \end{aligned} \right\} \quad (141a)$$

and, with

$$C_{21} + C_{22} + C_{23} = -\Delta_1 a_{11} \quad (141b)$$

d_2 is determined explicitly as

$$d_2 = (1+d_1)\Delta_1 \quad (141c)$$

(Note that this result can be obtained directly from expanding equation (117) and using (112) and (118). When Δ_2 is known, one similarly obtains the further result that $d_3 = (1+d_1)(\Delta_2 + \Delta_1^2/2)$.) For a given k , the quantity d_1 in equations (141) is known from the first outer solution. The value of Δ_1 must be obtained from the first inner solution. For use in the matching, the constants in equation (140b) are found to be

$$\left. \begin{aligned} a_{20} &= -a_{11}\Delta_1, & a_{21} &= (2a_{11} - 2a_{12})\Delta_1 \\ a_{22} &= (a_{11} - 3a_{12})\Delta_1, & a_{23} &= (1/3)(7a_{11} - 13a_{12})\Delta_1 \end{aligned} \right\} \quad (142)$$

The same procedure that was used to obtain equation (130) from one term of ψ^0 now leads to two terms of $\tilde{\psi}^0$ corresponding to two terms of ψ^0 :

$$\begin{aligned} \tilde{\psi}^0 &= \sin^2 \phi \{ (a_{11}\tilde{\xi} + a_{12}\tilde{\xi}^2 + a_{13}\tilde{\xi}^3 + \dots) \\ &\quad + \epsilon [(2a_{12}\Delta_1 + a_{21})\tilde{\xi} + (3a_{13}\Delta_1 + a_{22})\tilde{\xi}^2 + \dots] + O(\epsilon^2) \} \end{aligned} \quad (143)$$

Upon comparing equation (143) with (119c) we see that $\tilde{\psi}_2$ is the factor multiplying ϵ in (143).

Outer boundary conditions for the inner problem.- Equation (143) is used to find the outer boundary conditions for the inner problem. The terms to order ϵ in equation (143) constitute ψ^{20} (cf. eq. (73)). From equations (112), (119a), and (131a) we have

$$\tilde{\phi} = \phi, \quad \tilde{\zeta} = \epsilon(Z - \Delta_1) - \epsilon^2 \Delta_2 - \epsilon^3 \Delta_3 - \dots \quad (144)$$

which, when substituted into ψ^{20} , lead to

$$\begin{aligned} (\psi^{20})^{2i} = \sin^2 \phi \{ & \epsilon a_{11}(Z - \Delta_1) + \epsilon^2 [a_{12}(Z - \Delta_1)^2 \\ & + (a_{21} + 2a_{12}\Delta_1)(Z - \Delta_1) - a_{11}\Delta_2] \} \end{aligned} \quad (145)$$

as defined in the discussion following equation (74). The matching principle (76) requires that

$$\psi^{2i} \sim (\psi^{20})^{2i} + \exp \quad \text{as } Z \rightarrow \infty \quad (146)$$

where

$$\psi^{2i} = \epsilon \Psi_1 + \epsilon^2 \Psi_2 \quad (147)$$

(cf. eq. (74)). From equations (145), (146), and (147) we then find the outer boundary conditions for Ψ_1 and Ψ_2 :

$$\Psi_1 \sim \sin^2 \phi [a_{11}(Z - \Delta_1)] + \exp \quad \text{as } Z \rightarrow \infty \quad (148a)$$

$$\Psi_2 \sim \sin^2 \phi [a_{12}(Z - \Delta_1)^2 + (a_{21} + 2a_{12}\Delta_1)(Z - \Delta_1) - a_{11}\Delta_2] + \exp \quad \text{as } Z \rightarrow \infty \quad (148b)$$

Solution of the first- and second-order inner problems.- Now substitution of equation (133) into the partial differential equation for $\Psi(\phi, Z, k, l/R)$ found above (see discussion following eq. (136b)) and into the conditions (135), and letting $\epsilon \rightarrow 0$ successively, leads to the following problems for the first two terms in the inner expansion (133):

$$\left. \begin{aligned} \frac{\partial}{\partial Z} \left[\Psi_{1Z} \Psi_{1\phi Z} - \Psi_{1\phi} \Psi_{1ZZ} - (\cot \phi) \Psi_{1Z}^2 - (\sin \phi) \Psi_{1ZZZ} \right] &= 0 \\ \Psi_1 = \Psi_{1Z} &= 0 \quad \text{at } Z = 0 \end{aligned} \right\} \quad (149)$$

with equation (148a) as the outer condition, and

$$\left. \begin{aligned}
& \frac{\partial}{\partial Z} \left[\Psi_{1Z} \Psi_{2\varphi Z} + \Psi_{1\varphi Z} \Psi_{2Z} - \Psi_{1\varphi} \Psi_{2ZZ} - \Psi_{1ZZ} \Psi_{2\varphi} - 2(\cot \varphi) \Psi_{1Z} \Psi_{2Z} - (\sin \varphi) \Psi_{2ZZZ} \right] \\
& = \Psi_{1Z} \Psi_{1Z\varphi} - 2(\cot \varphi) \Psi_{1Z}^2 - 5\Psi_{1\varphi} \Psi_{1ZZ} - \sin \varphi \left[6\Psi_{1ZZZ} - Z\Psi_{1ZZZZ} \right] \\
& \Psi_2 = \Psi_{2Z} = 0 \quad \text{at } Z = 0
\end{aligned} \right\} (150)$$

with equation (148b) as the outer condition. The problem for Ψ_1 is, of course, the same as would be obtained by Prandtl's first-order boundary-layer theory.

As discussed above (prior to eq. (138)), in solving the problem for Ψ_1 we will not assume a priori that $\Delta_1 = \text{constant}$ (not varying with φ), but will use the equations that determine the form of Δ_1 and then observe the significance of requiring $\Delta_1 = \text{constant}$. The bracketed quantity in equation (149) must be a function only of φ , say $\Phi(\varphi)$. Differentiation of condition (148a) to obtain asymptotic expressions for the various terms in $\Phi(\varphi)$ then leads to

$$\Phi(\varphi) = a_{11}^2 \sin^4 \varphi \cot \varphi \quad (151)$$

Now assume

$$\Psi_1 = \sqrt{2\xi} F_1(\eta) \quad (152)$$

where

$$\left. \begin{aligned}
\xi &= \xi(\varphi) \equiv a_{11} \int_0^\varphi \sin^3 \varphi \, d\varphi \\
&= a_{11} \left[1 - \cos \varphi - \frac{1}{3} (1 - \cos^3 \varphi) \right] \\
&= \frac{a_{11} \sin^4 \varphi}{4} \left[1 + \frac{1}{3} \sin^2 \varphi + O(\sin^4 \varphi) \right]
\end{aligned} \right\} (153a)$$

and where

$$\eta = \eta(\varphi, Z) \equiv \lambda Z \quad (153b)$$

$$\lambda = \lambda(\sin \varphi) \equiv \frac{a_{11} \sin^2 \varphi}{\sqrt{2\xi}} = \sqrt{2a_{11}} \left[1 - \frac{1}{6} \sin^2 \varphi + O(\sin^4 \varphi) \right] \quad (153c)$$

Also define a function A by

$$\left. \begin{aligned} A &= A(\sin \varphi) \equiv \frac{2\xi \cos \varphi}{a_{11} \sin^4 \varphi} \\ &= \frac{1}{2} \left[1 - \frac{1}{6} \sin^2 \varphi + O(\sin^4 \varphi) \right] \end{aligned} \right\} \quad (154)$$

The problem for Ψ_1 given by equations (148a) and (149) then becomes

$$F_1''' + F_1 F_1'' = A[(F_1')^2 - 1] \quad (155a)$$

$$F_1(0) = F_1'(0) = 0 \quad (155b)$$

$$F_1 \sim \eta - \lambda \Delta_1 + \exp \quad \text{as } \eta \rightarrow \infty \quad (155c)$$

or

$$F_1' \sim 1 + \exp \quad \text{as } \eta \rightarrow \infty \quad (155c')$$

where $()'$ here denotes differentiation with respect to η . A local similarity solution is obtained by assuming A to be a constant appropriate to the particular value of φ (cf. Stine and Wanlass, ref. 50, or Lees, ref. 51). In particular, for the region near the stagnation streamline, neglect of $(1/6)\sin^2 \varphi$ in comparison to unity in equation (154) leads to a locally similar solution with $A = 1/2$. Further, neglect of $(1/6)\sin^2 \varphi$ in comparison to unity in equation (153c) then leads to a constant value for Δ_1 near $\varphi = 0$ with $\lambda = \sqrt{2a_{11}}$. For consistency, that is, in order to neglect all quantities of $O(\sin^2 \varphi)$ as $\varphi \rightarrow 0$, we can also neglect terms as small as $(1/3)\sin^2 \varphi$ in equation (153a), so that we now take

$$\xi = (1/4)a_{11} \sin^4 \varphi; \quad \lambda = \sqrt{2a_{11}}; \quad A = 1/2 \quad (156)$$

in the determination of Ψ_1 by equations (151), (153b), and (155).

We see that the assumption of local spherical symmetry in the outer solution and further local similarity in the inner solution require neglect of $O(\sin^2 \varphi)$ in comparison to unity in some quantities in the inner solution (cf. discussion by Kao, ref. 46, p. 1893). This then yields constant Δ_1 which, further, will allow the condition of local spherical symmetry in the outer solution to continue to second order.

Equation (155a) is equivalent to the equation derived by Falkner and Skan (ref. 52). The numerical solution corresponding to $A = 1/2$ was first given by Homann in 1936 (see ref. 53). More general solutions were studied

by Hartree (ref. 54). A convenient tabulation of the solution for $A = 1/2$ is given in reference 55 (table V.3, p. 237) where, for example,

$$\left. \begin{aligned} F_1''(0) &= 0.9278 \\ \text{and} \\ \beta_1 &\equiv \lim_{\eta \rightarrow \infty} (\eta - F_1) = 0.8046 \end{aligned} \right\} \quad (157)$$

From this solution

$$\Delta_1 \equiv \beta_1/\lambda = 0.8046/\sqrt{2a_{11}} \quad (158)$$

where a_{11} , depending on k , is given by equation (127b). This agrees with Van Dyke's evaluation (ref. 30) of the first-order displacement thickness (there referred to as the displacement thickness) at the stagnation point for the special case $k = 1/6$ for which $a_{11} = 2/3$.

The second-order boundary-layer problem is given by equations (150) with the outer condition (148b). A local similarity solution near $\varphi = 0$ is obtained as

$$\Psi_2 = (1/\lambda) \sqrt{2\xi} F_2(\eta) \quad (159)$$

(with neglect of $O(\sin^2 \varphi)$ in comparison to unity, as in the problem for Ψ_1) where ξ and λ are given by equation (156) and $\eta = \lambda Z$. The resulting ordinary differential equation is

$$(d/d\eta)[F_2''' + F_1 F_2'' - F_1' F_2' + F_1'' F_2] = 5F_1 F_1'' + 6F_1''' - \eta F_1'''' \quad (160a)$$

with the conditions

$$F_2(0) = F_2'(0) \quad (160b)$$

$$F_2 \sim (a_{12}/a_{11})(\eta - \beta_1)^2 + 2\beta_1(\eta - \beta_1) - \beta_2 + \exp \quad \text{as } \eta \rightarrow \infty \quad (160c)$$

from which also

$$F_2' \sim (2a_{12}/a_{11})(\eta - \beta_1) + 2\beta_1 + \exp \quad \text{as } \eta \rightarrow \infty \quad (160d)$$

$$F_2'' \sim 2a_{12}/a_{11} + \exp \quad \text{as } \eta \rightarrow \infty \quad (160e)$$

and where

$$\beta_2 \equiv 2a_{11}\Delta_2 \quad (161)$$

The quantity Δ_2 is the second-order term for displacement in equation (112). With use of equation (155a) with $A = 1/2$, equation (160a) can be integrated to

$$L\{F_2\} = (5/3)(F_1 F_1' - \eta) + (11/3)F_1'' - \eta F_1''' + C_1 \quad (162)$$

where L is the linear differential operator defined by

$$L\{F_2\} = \left(\frac{d^3}{d\eta^3} + F_1 \frac{d^2}{d\eta^2} - F_1' \frac{d}{d\eta} + F_1'' \right) F_2 \quad (163)$$

and where C_1 is a constant of integration determined by the asymptotic boundary conditions to be

$$C_1 = -(1/3)\beta_1 \quad (164)$$

The problem for F_2 is most easily solved by first splitting F_2 into two parts as follows. Let

$$F_2(\eta) = F_{21}(\eta) + F_2''(0)F_{22}(\eta) \quad (165)$$

where $F_{21}(\eta)$ is defined as the solution to the problem:

$$L\{F_{21}\} = (5/3)(F_1 F_1' - \eta) + (11/3)F_1'' - \eta F_1''' + C_1 \quad (166a)$$

$$F_{21}(0) = F_{21}'(0) = F_{21}''(0) = 0 \quad (166b)$$

Then the problem for $F_{22}(\eta)$ becomes

$$L\{F_{22}\} = 0 \quad (167a)$$

$$F_{22}(0) = F_{22}'(0) = 0 ; \quad F_{22}''(0) = 1 \quad (167b)$$

It is easily found by asymptotic integration that equations (166a) and (167a) have the asymptotic solutions

$$\left. \begin{aligned} F_{21} &\sim K_2(\eta - \beta_1)^2 + K_1(\eta - \beta_1) + K_3 + \exp & \text{as } \eta \rightarrow \infty \\ F_{22} &\sim K_4(\eta - \beta_1)^2 + K_5 + \exp & \text{as } \eta \rightarrow \infty \end{aligned} \right\} \quad (168)$$

where the constants K_2, K_3, K_4 , and K_5 are determined by the initial conditions (166b) and (167b), and where $K_1 = 2\beta_1$. The constants can be obtained by numerical integration of equations (166) and (167) and asymptotic evaluation of the limits (found from (168)):

$$\left. \begin{aligned} K_2 &= (1/2)F_{21}''(\infty) , & K_4 &= (1/2)F_{22}''(\infty) \\ K_3 &= \lim_{\eta \rightarrow \infty} [F_{21}(\eta) - 2\beta_1(\eta - \beta_1) - (1/2)(\eta - \beta_1)^2 F_{21}''(\eta)] \\ K_5 &= \lim_{\eta \rightarrow \infty} [F_{22}(\eta) - (1/2)(\eta - \beta_1)^2 F_{22}''(\eta)] \end{aligned} \right\} \quad (169)$$

Then it is found from conditions (160c), (160d), and (160e) that

$$\left. \begin{aligned} F_2''(0) &= \lim_{\eta \rightarrow \infty} \left[\frac{2a_{12}/a_{11} - F_{21}''(\eta)}{F_{22}''(\eta)} \right] = \frac{a_{12}/a_{11} - K_2}{K_4} \\ \text{and} \\ \beta_2 &= -K_3 - K_5 F_2''(0) \end{aligned} \right\} \quad (170)$$

Note from equations (167) that the solutions for F_{21} and F_{22} depend only on prior solution of the first-order boundary-layer problem (for F_1), and so are independent of the external conditions. For an arbitrary k , numerical computation of the constants K_2 , K_3 , K_4 , and K_5 then leads to:

$$\left. \begin{aligned} F_2''(0) &= 0.649056(2a_{12}/a_{11}) - 0.774713 \\ \beta_2 &= 1.3796 - 0.701954(2a_{12}/a_{11}) \end{aligned} \right\} \quad (171)$$

where a_{11} and a_{12} are given by equations (127) and where d_1 is the root of equation (126b).

Determination and matching of pressure. - As some difficulty in the past has been associated with the determination of the second-order pressure, consider now the pressure and the matching of pressure. Corresponding to ψ^0 we have

$$\bar{p}^0 = \bar{p}^0(\varphi, \zeta, k, 1/R) = p_1(\varphi, \zeta, k) + \epsilon p_2(\varphi, \zeta, k) + \epsilon^2 p_3(\varphi, \zeta, k) + \dots \quad (172a)$$

where \bar{p}^0 is the assumed analytic outer function representing \bar{p} asymptotically in the outer region (as $1/R \rightarrow 0$ with ζ fixed). Corresponding to $\tilde{\psi}^0$, we have

$$\begin{aligned} \bar{p}^0 &= \tilde{p}^0 = \tilde{p}^0(\tilde{\varphi}, \tilde{\zeta}, k, 1/R) \\ &= \tilde{p}_1(\tilde{\varphi}, \tilde{\zeta}, k) + \epsilon \tilde{p}_2(\tilde{\varphi}, \tilde{\zeta}, k) + \epsilon^2 \tilde{p}_3(\tilde{\varphi}, \tilde{\zeta}, k) + \dots \end{aligned} \quad (172b)$$

The inner function \bar{p}^1 is to represent \bar{p} asymptotically as $1/R \rightarrow 0$ with Z fixed. Thus, following the convention outlined above, define

$$\left. \begin{aligned} \sigma_p(1/R)P(\varphi, Z, k, 1/R) &\equiv \bar{p}^i(\varphi, Z, k, 1/R) \sim \bar{p}(\varphi, \zeta, k, 1/R) + \exp \\ &\text{as } 1/R \rightarrow 0 \text{ with } Z \text{ fixed} \end{aligned} \right\} \quad (173)$$

where

$$P = P_1(\varphi, Z, k) + \epsilon P_2(\varphi, Z, k) + \epsilon^2 P_3(\varphi, Z, k) + \dots \quad (174)$$

is assumed analytic with respect to ϵ at $\epsilon = 0$. The factor $\sigma_p(1/R)$ will be determined to make P_1 the first nonzero term in equation (174). The inner and outer pressure functions will be seen to match according to

$$\bar{p}^{ni} \sim (\bar{p}^{no})^{ni} + \exp \quad \text{as } Z \rightarrow \infty \quad (175)$$

as expected (cf. eq. (76)).

For determination of the terms in equations (172) and (174), equations (102) can be put in terms of ψ . The equations for \bar{p}^0 are then the corresponding equations in terms of ψ^0 and the equations for \bar{p}^1 are the corresponding equations in terms of ψ .

The first- and second-order terms of the pressure function in the outer problem are then to be found from:

$$k(\partial p_1 / \partial \varphi) = -\sin \varphi \cos \varphi e^{-4\zeta} [(f_1')^2 - 2f_1 f_1'' + 2f_1 f_1'] \quad (176a)$$

$$k(\partial p_1 / \partial \zeta) = -\sin^2 \varphi e^{-4\zeta} [2f_1 f_1' - (f_1')^2] - \cos^2 \varphi e^{-4\zeta} [4f_1 f_1' - 8f_1^2] \quad (176b)$$

and

$$k(\partial p_2 / \partial \varphi) = -2 \sin \varphi \cos \varphi e^{-4\zeta} [f_1' f_2' - f_1 f_2'' - f_1'' f_2 + (f_1 f_2)'] \quad (176c)$$

$$\begin{aligned} k(\partial p_2 / \partial \zeta) &= -2 \sin^2 \varphi e^{-4\zeta} [f_1 f_2' + f_1' f_2 - f_1' f_2'] \\ &\quad - 4 \cos^2 \varphi e^{-4\zeta} [(f_1 f_2)' - 4f_1 f_2] \end{aligned} \quad (176d)$$

Equations (176a) and (176c) may be integrated with respect to φ , then differentiated with respect to ζ and equated, respectively, to equations (176b) and (176d) to evaluate functions of integration, making use of the differential equations (124) and (139). The constants of integration are evaluated by the boundary conditions on f_1 and f_2 at (transferred to) ζ_{s1} . The results are

$$\begin{aligned} -kp_1 &= \sin^2 \varphi e^{-4\zeta} \left[\frac{1}{2} (f_1')^2 - f_1 f_1'' + f_1 f_1' \right] \\ &\quad + 2e^{-4\zeta} f_1^2 + \frac{1}{2} k^2 - k \end{aligned} \quad (177a)$$

$$-kP_2 = \sin^2 \varphi e^{-4\zeta} [f_1' f_2' - f_1 f_2'' - f_1'' f_2 + (f_1 f_2)'] + 4e^{-4\zeta} f_1 f_2 \quad (177b)$$

In the equations for the inner problem it is found that, for nonzero P_1 , $\sigma_p(1/R)$ must be $O(1)$ as $1/R \rightarrow 0$; thus, we take $\sigma_p = 1$. The equations for P_1 are then

$$\left. \begin{aligned} k \sin^2 \varphi [\partial P_1 / \partial (\sin \varphi)] &= -\Phi(\varphi) / \cos \varphi = a_{11}^2 \sin^3 \varphi \\ kP_{1Z} &= 0 \end{aligned} \right\} \quad (178)$$

where $\Phi(\varphi)$ is the bracketed quantity in equation (149), with the result

$$kP_1 = -(1/2)a_{11}^2 \sin^2 \varphi + \text{constant}$$

Since $k(\bar{p}^{10})^{11} = -(1/2)a_{11}^2 \sin^2 \varphi + k - (1/2)k^2$ and $\bar{p}^{11} \equiv P_1$, the matching rule (175) is satisfied for $n = 1$ by

$$kP_1 = -(1/2)a_{11}^2 \sin^2 \varphi + k - (1/2)k^2 \quad (179)$$

The equations for P_2 are (with neglect of $O(\sin^2 \varphi)$ in comparison to unity and with use of equations (155a) and (162)):

$$\left. \begin{aligned} k[\partial P_2 / \partial (\sin \varphi)] &= \sqrt{2} a_{11}^{3/2} (\sin \varphi) [(2/3)(F_1'' + F_1 F_1') + (1/3)\eta + C_1] \\ k(\partial P_2 / \partial \eta) &= (1/\sqrt{2}) a_{11}^{3/2} (\sin^2 \varphi) (F_1')^2 \end{aligned} \right\} \quad (180)$$

which give

$$kP_2 = (1/\sqrt{2}) a_{11}^{3/2} (\sin^2 \varphi) [(2/3)(F_1'' + F_1 F_1') + (1/3)\eta + C_1] + \text{constant}^0 \quad (181)$$

where C_1 is given by equation (164). Since

$$\begin{aligned} k(\bar{p}^{20})^{21} &= [-(1/2)a_{11}^2 \sin^2 \varphi + k - (1/2)k^2] + \epsilon \sin^2 \varphi [a_{11}^2 Z \\ &\quad - a_{11} a_{21} + 2a_{12} a_{20} - a_{11} a_{20}] \end{aligned} \quad (182)$$

and $\bar{p}^{21} \equiv P_1 + \epsilon P_2$, it is found that equation (175) is satisfied exactly for $n = 2$ only if the constant of integration in equation (181) is zero.

The body surface pressure is given by

$$\begin{aligned} \left(\frac{p - p_\infty}{\rho_\infty V_\infty^2} \right)_b &\equiv \bar{p}_b \sim \bar{p}^1 \left(\varphi, 0, k, \frac{1}{R} \right) = P_1(\varphi, 0, k) + \epsilon P_2(\varphi, 0, k) + \dots \\ &= \left(1 - \frac{k}{2} - \frac{a_{11}^2}{2k} \sin^2 \varphi \right) + \frac{\epsilon (a_{11})^{3/2}}{\sqrt{2} k} \left[\frac{2}{3} F_1''(0) + C_1 \right] \sin^2 \varphi + \dots \end{aligned} \quad (183)$$

Results for velocity and skin friction. - The dimensionless velocity components u^o and v^o corresponding to the outer function ψ^o are defined by

$$u^o \equiv \frac{e^{-2\zeta} \psi_{\zeta}^o}{\sin \varphi} = (\sin \varphi) e^{-2\zeta} [f_1'(\zeta) + \epsilon f_2'(\zeta) + O(\epsilon^2)] \quad (184a)$$

$$v^o \equiv \frac{-e^{-2\zeta} \psi_{\varphi}^o}{\sin \varphi} = -(2 \cos \varphi) e^{-2\zeta} [f_1(\zeta) + \epsilon f_2(\zeta) + O(\epsilon^2)] \quad (184b)$$

The velocity components corresponding to the inner function ψ^i are

$$u^i \equiv \frac{e^{-2\epsilon Z} \psi_Z}{\sin \varphi} = \sin \varphi \left\{ a_{11} F_1'(\eta) + \epsilon \sqrt{\frac{a_{11}}{2}} [F_2'(\eta) - 2\eta F_1'(\eta)] + O(\epsilon^2) \right\} \quad (184c)$$

$$v^i \equiv \frac{-e^{-2\epsilon Z} \epsilon \psi_{\varphi}}{\sin \varphi} = -\epsilon (\cos \varphi) \left\{ \sqrt{2a_{11}} F_1(\eta) + \epsilon [F_2(\eta) - 2\eta F_1(\eta)] + O(\epsilon^2) \right\} \quad (184d)$$

It is easily shown that the inner and outer functions representing both u and v (eqs. (184)) satisfy the matching rules (cf. eq. (76)):

$$\left. \begin{aligned} u^{ni} &\sim (u^{no})^{ni} + \exp & \text{as } Z \rightarrow \infty \\ v^{ni} &\sim (v^{no})^{ni} + \exp & \text{as } Z \rightarrow \infty \end{aligned} \right\} \quad (185)$$

for $n = 1$ and $n = 2$. The skin-friction coefficient is

$$C_f \equiv \frac{\tau_b}{(1/2) \rho_{\infty} V_{\infty}^2} = \frac{2\mu_b}{\rho_{\infty} V_{\infty} r_b} \left(r \frac{\partial u}{\partial r} \right)_{r=r_b} = \frac{2}{kR \sin \varphi} (\psi_{\zeta\zeta} - 2\psi_{\zeta})_{\zeta=0} \quad (186a)$$

$$= \frac{2\epsilon}{k \sin \varphi} [\Psi_{1ZZ}(\varphi, 0, k) + \epsilon \Psi_{2ZZ}(\varphi, 0, k) + O(\epsilon^2)] \quad (186b)$$

$$= \epsilon \sin \varphi \left[\frac{(2a_{11})^{3/2}}{k} F_1''(0) + \epsilon \frac{2a_{11}}{k} F_2''(0) + O(\epsilon^2) \right] \quad (186c)$$

$$\equiv \epsilon \sin \varphi [\bar{C}_{f_1} + \epsilon \bar{C}_{f_2} + O(\epsilon^2)] \quad (186d)$$

Computed values of \bar{C}_{f_1} and \bar{C}_{f_2} are listed in the following tabulation. Corresponding values of d_1 , d_2^2 , Δ_1 , and Δ_2 in equations (118) and (112) are also given.

k	\bar{C}_{f1}	\bar{C}_{f2}	d_1	d_2	Δ_1	Δ_2
1/6	8.56954	24.9568	0.118034	0.77900	0.696759	-2.12408
1/10	9.76336	90.0171	.073052	.84874	.790954	-8.57017
1/20	11.6143	432.552	.038646	.97686	.940511	-42.9388

Results are shown in figures 4 to 8. The dimensionless magnified boundary layer displacement is plotted to first and second order in figure 6 (cf. eq. (114)):

$$\Delta' \equiv \frac{y\delta}{\epsilon} \equiv \frac{e^{\epsilon\Delta} - 1}{\epsilon} = \Delta_1 + \epsilon \left(\Delta_2 + \frac{1}{2} \Delta_1^2 \right) + \dots \quad (187)$$

where $r_b y_\delta$ is the actual displacement distance corresponding to the displacement $\delta = \epsilon\Delta$ of the ξ coordinate. These results are discussed in the next section.

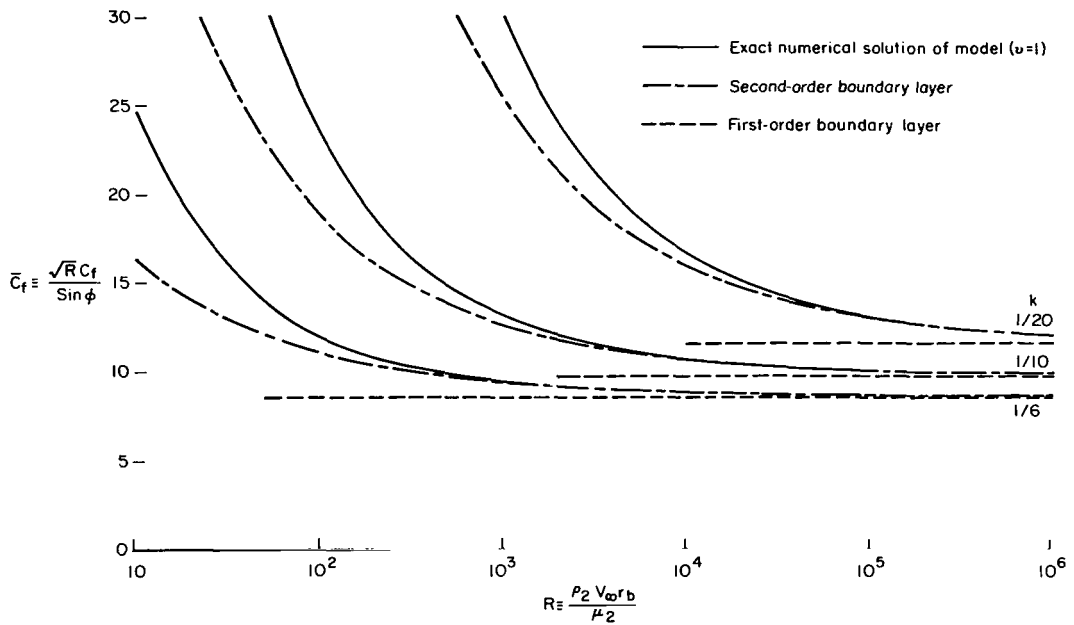


Figure 4.- Skin friction.

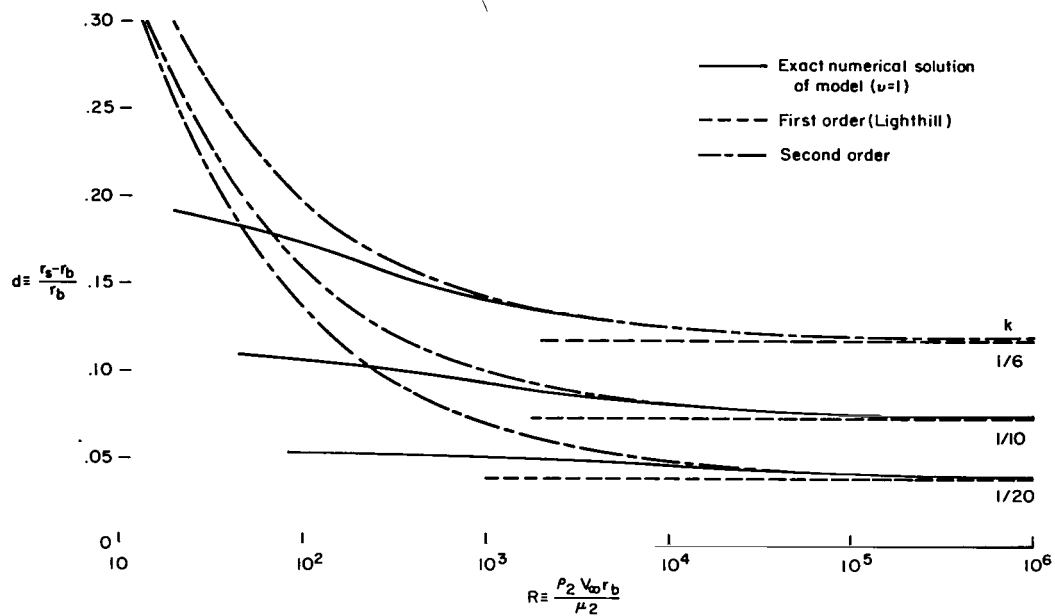


Figure 5.- Shock standoff distance.

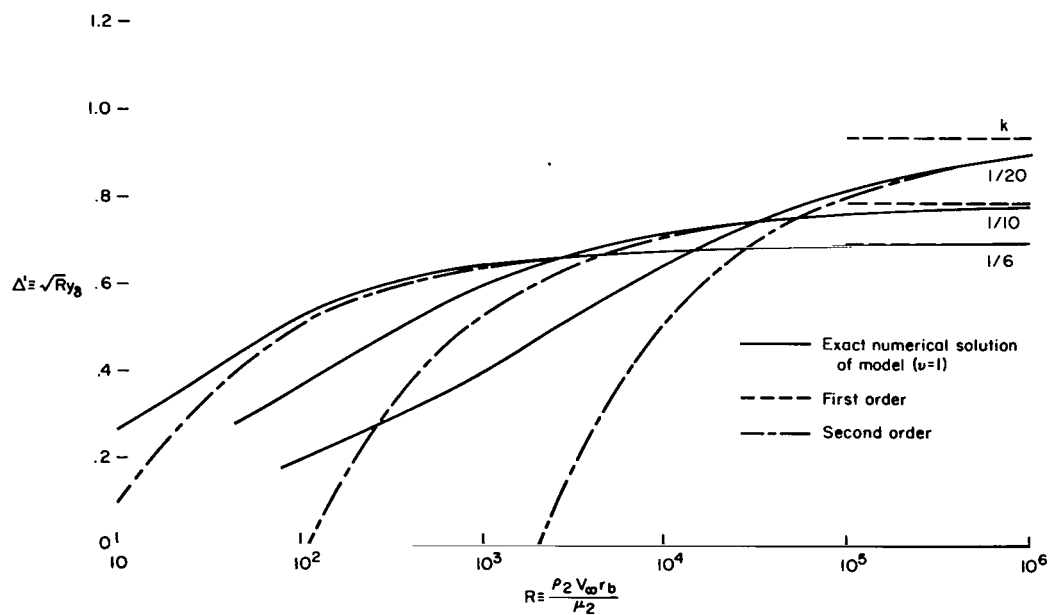


Figure 6.- Boundary-layer displacement.

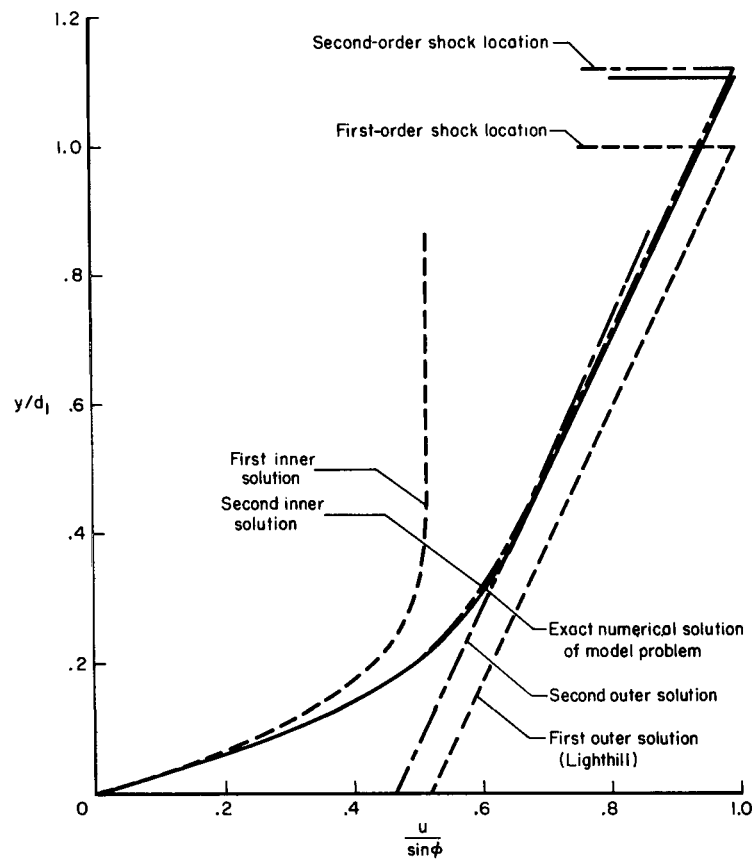


Figure 7.- Example of tangential velocity profile,
 $k = 0.1$, $R = 9.251 \times 10^3$.

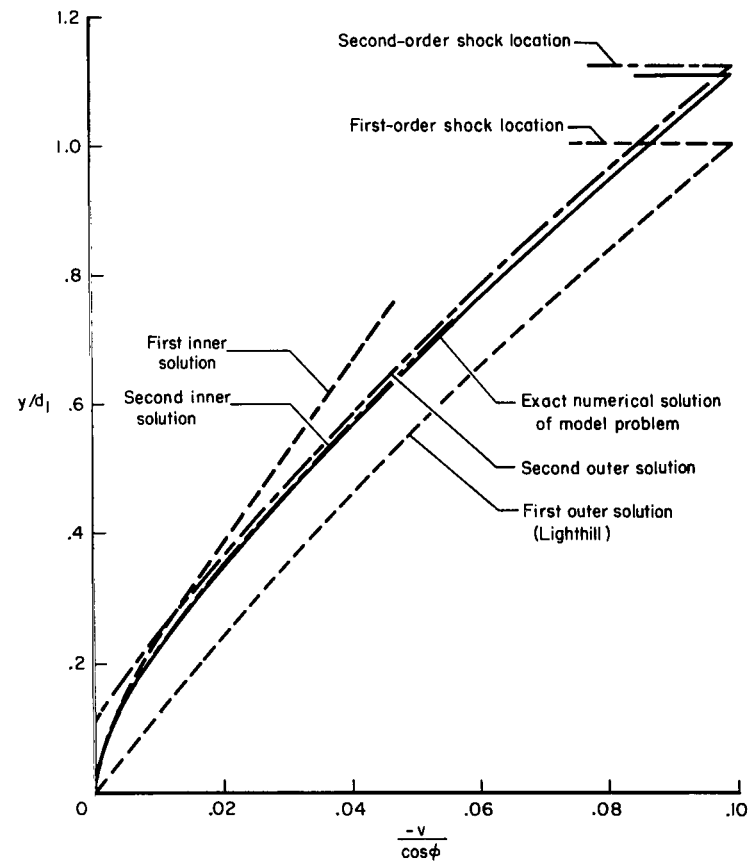


Figure 8.- Example of normal velocity profile,
 $k = 0.1$, $R = 0.9251 \times 10^4$.

Comparison With a Numerical Solution of the Model Problem

The results of applying the asymptotic-expansion method to the model for viscous hypersonic flow described above can be compared with a numerical local similarity solution of the same model problem. The equations to be solved are (103), (104), and (105). This problem will be seen to possess the property of local similarity near $\varphi = 0$. The solution obtained is similar to Probstein's (ref. 48), but for greater simplicity the viscosity has been taken to be constant here.

Consider the following assumed separation of variables:

$$\psi = \frac{\sin^2 \varphi e^{\xi} G(\xi)}{R} \quad (188)$$

(The presence of the factors $1/R$ and e^{ξ} in equation (188) will lead to elimination of those same factors from the differential equation.) For convenience, define a parameter

$$N \equiv kR(1 + d) = \frac{\rho_{\infty} V_{\infty} r_s}{\mu_2} \quad (189)$$

and a new variable

$$\bar{\xi} \equiv \xi - \xi_s \equiv \log \left(\frac{1 + y}{1 + d} \right) \quad (190)$$

Substitution of equation (188) into (103), (104), and (105) gives the differential equation

$$G'''' - 2G''' - 5G'' + 6G' = -2(\cos \varphi)G(G'''' - 2G''' - 5G'' + 6G') \quad (191)$$

where $()'$ denotes differentiation with respect to either ξ or $\bar{\xi}$, and the boundary conditions:

At $\xi = 0$, $\bar{\xi} = \bar{\xi}_0 \equiv -\xi_s \equiv -\log(1 + d)$:

$$G = 0 \quad (192a)$$

$$G' = 0 \quad (192b)$$

and at $\xi = \xi_s \equiv \log(1 + d)$, $\bar{\xi} = 0$:

$$G = (1/2)N \quad (192c)$$

$$G' = N(1/k - 1/2) \quad (192d)$$

and

$$G'' = \frac{-\nu}{\cos \varphi} \left(\frac{G'''}{N} - \frac{3}{k} + \frac{5}{2} \right) + N \left(\frac{1}{k^2} - \frac{3}{k} + \frac{5}{2} \right) \quad (192e)$$

One sees that the assumed form, equation (188), did not lead to true separation of variables, but that the well-known condition of local similarity near $\varphi = 0$ is obtained by taking $\cos \varphi$ equal to unity in equations (191) and (192e). Then the resulting fourth-order ordinary differential equation can be solved with the five boundary conditions, equation (192), one of which is required to determine the unknown d . The two input parameters (to be specified) are N and k . The fourth-order differential equation (191) with $\cos \varphi = 1$ can be integrated once to obtain

$$G''' - 2G'' - 5G' + 6G = Ce^{\bar{\xi}} \quad (193)$$

where

$$g \equiv - \int_{\xi_s}^{\xi} 2G \, d\xi = - \int_0^{\bar{\xi}} 2G \, d\bar{\xi} \quad (194)$$

and where

$$C \equiv (G''' - 2G'' - 5G' + 6G)_{\xi=\xi_s} \quad (195)$$

Since three conditions are specified at the shock and only two at the body, it is most convenient to start integrating at the shock, where a fourth condition (the value of G''' or C), must be guessed and iterated. The problem of the unknown shock location is eliminated by using $\bar{\xi}$ as the independent variable, which has the value zero at the shock. Equations (192e) and (195) provide two simultaneous equations to solve for G'' and G''' at $\bar{\xi} = 0$ in terms of C if C is to be specified and iterated upon.

To solve the problem numerically, one estimates a value for C , integrates equation (193) with decreasing $\bar{\xi}$, and stops when $G' = 0$. By an appropriate iteration scheme, one adjusts the value of C until condition (192a) is satisfied at the same value of $\bar{\xi}$ where (192b) is satisfied. That location is then denoted as $\bar{\xi} = \bar{\xi}_0 \equiv -\xi_s \equiv -\log(1+d)$, from which ξ_s and d are determined. It is convenient to use the form (193), rather than (191), and to iterate on values of C rather than on $G'''(\bar{\xi} = 0)$ because an initial estimate for C is more easily obtained. Furthermore, when N is large, C is very small and the iteration becomes difficult. The difficulty is remedied by estimating (from boundary-layer theory) a value of $\bar{\xi}$, say $\bar{\xi}_1$, where the right side of equation (193) begins to become significant. In that case, one keeps the right side of equation (193) at zero until $\bar{\xi}_1$ is reached, and then replaces the right side by $qe^{\bar{\xi}}$, where q is some specified small number and

$$\bar{g} \equiv \int_{\bar{\xi}_1}^{\bar{\xi}} -2G \, d\bar{\xi}$$

Since C is taken to be zero for the first part of the integration in this case, one iterates instead on the value of $\bar{\xi}_1$, corresponding to a specified q , until the conditions at the body are satisfied. Different small values of

q are used for each case, and the results compared, to ensure that q is small enough that the solution obtained is not affected, to within the desired numerical accuracy, by this procedure.

The dimensionless velocity components (see eqs. (99)) are given in this solution by

$$\left. \begin{aligned} u &= (1/R)(\sin \varphi)e^{-\zeta}(G' + G) \\ v &= -2(1/R)(\cos \varphi)e^{-\zeta}G \end{aligned} \right\} \quad (196)$$

and the skin-friction coefficient, defined in equation (186a), is

$$C_f = (2/kR^2)(\sin \varphi)G''(\zeta = 0) \quad (197)$$

The dimensionless pressure gradient along the body surface is (with the local similarity condition):

$$\left[\frac{\partial \bar{p}}{\partial (\sin \varphi)} \right]_b = \frac{(\sin \varphi)G'''(\zeta = 0)}{kR^2} \quad (198)$$

To find the complete viscous displacement, one needs the complete outer solution for flow over a displaced surface at $\zeta = \delta$ with no boundary layer present. That is, for the outer solution, the no-slip condition is replaced by the condition that there be no boundary layer, in line with the concept that $\zeta = \delta$ is the frictionless surface that would produce the flow field that corresponds to the outer solution, which is approached exponentially by the exact solution. The complete outer solution is $\psi^0 = (\sin^2 \varphi)f^0(\zeta)$, where f^0 is the solution $e^{\zeta}G(\zeta)/R$ that satisfies equations (191) and (192) but with equations (192a) and (192b) replaced by $f^0 = 0$ at $\zeta = \delta$ and the condition that no rapid (exponential) variations with ζ near $\zeta = \delta$ are permitted. The solution for f^0 is found to have the same form as equation (125), but the constants are found from the complete boundary conditions at the shock and thus depend on R for $v = 1$. The condition $f^0 = 0$ at $\zeta = \delta$ then gives the following equation (refer to p. 42):

$$3(1 - k)^2 + 5(4k - 1)(1 + d^*)^2 + 2(1 - k)(1 - 6k)(1 + d^*)^5 + \frac{20v}{R(1 + d)} \left[\left(1 + \frac{1}{2}k\right)(1 + d^*)^2 - (1 - k)(1 - 6k)(1 + d^*)^5 \right] = 0 \quad (199)$$

For a given case, N is specified and d is determined, as discussed above, so d^* is determined by this equation. Then δ is determined from equation (117) as

$$\delta = \log \left(\frac{1 + d}{1 + d^*} \right) \quad (200)$$

and $\Delta' \equiv R^{1/2}y_\delta$ is found from equation (116).

Although the calculations were made for both $\nu = 0$ and $\nu = 1$, only those for $\nu = 1$ are shown in figures 4 to 8. The values of C_f for $\nu = 0$ and $\nu = 1$ differ very little for $R > 500$. The differences are much less than the differences shown between the second-order solution and the exact numerical solution for $\nu = 1$. Similarly, the differences in the values of d for $\nu = 0$ and $\nu = 1$ were very small for $R > 100$.

In the plot of $\sqrt{R} C_f / \sin \phi$ (fig. 4), first-order boundary-layer theory gives a constant value, shown as a dashed line for each k . Note in particular that the second-order solution follows closely the exact numerical solution of the model for fairly low R when k is not too small. However, apparently the asymptotic convergence to the exact solution of the model problem is slowed when k becomes small. For example, for $k = 1/6$, the first-order boundary-layer solution for C_f at $R = 1000$ is 9.8 percent low, and the second-order solution corrects that error to a value less than 1.6 percent low. For $k = 1/10$, the corresponding negative errors are 26 percent for first order and 4.6 percent for second order. For $k = 1/20$, those negative errors at $R = 1000$ are 61 percent for first order, and 16 percent for second order. The percent errors increase as k becomes small. This effect of small k slowing the asymptotic convergence can also be observed from the numbers given in the table following equation (186d). As k becomes small, the numerical factor of the second term \bar{C}_{f_2} becomes large in comparison to that of the first term \bar{C}_{f_1} . This can be shown to be the effect of increasing vorticity interaction; as k becomes small, the vorticity in the external flow (outer solution) becomes comparable to the vorticity generated at the body surface.

Similarly, the plots of shock standoff distance and boundary-layer displacement, and the tabulated numerical factors following equation (186d) exhibit the same slowing of convergence as k becomes small. Note in particular that the second-order boundary-layer displacement agrees well with the exact solution of the model problem for very low R when $k = 1/6$.

Figures 7 and 8 illustrate the substantial improvement of second-order theory in describing the u and v velocity profiles. First-order boundary-layer theory for the tangential velocity u goes asymptotic to a constant value equal to the first-order inviscid surface speed. For $k = 1/10$ and $R = 9.251 \times 10^3$, the second-order outer and inner solutions for u and v agree almost exactly with the exact numerical solution of the model with the inner solution agreeing almost exactly near $y = 0$, the outer agreeing more closely outside the boundary layer.

CONCLUDING REMARKS

It has been shown how a new approach to constructing inner and outer expansions allows the derivation and use of a stronger form of the rule for matching the expansions in some singular perturbation problems of the boundary-layer type. By use of this "displacement matching," all displacement effects are retained explicitly, and explicit asymptotic outer boundary

conditions for the inner function are provided. Application of this method to viscous hypersonic flow over a blunt body has led to an investigation of the role of boundary-layer displacement, including higher-order displacements, not heretofore considered. In particular, second-order displacement of the boundary layer has been calculated, and its relationship to the shock standoff distance indicated. The outer and inner expansions of all functions and derivatives of the solution match according to the same stronger form of the matching principle. Comparison of the second-order solution, including second-order viscous displacement, with an exact numerical solution of the model problem has exhibited very close agreement for Reynolds number only moderately large, and has exhibited a slowing of the asymptotic convergence as the shock density ratio k becomes small, indicating the increasing effects of vorticity interaction.

Ames Research Center

National Aeronautics and Space Administration

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APPENDIX

PRINCIPAL NOTATION

A	see equations (154) and (156)
a_{nm}	coefficients in expansions; see equations (125c), (127), (140b), and (142)
C	constant of integration first appearing in equation (193)
C_1	constant of integration first appearing in equation (162); evaluated in (164)
C_f	skin-friction coefficient, equations (186)
\bar{C}_{fn}	constants defined in equations (186)
D^*	domain of independent variables defined on pages 17 and 18 or on page 34
D^i	domain of magnified independent variables where the magnified inner function is analytic with respect to ϵ_i (see eq. (60c)) or where the series for F converges in equations (88)
D^o	domain of independent variables where outer function f^o is analytic with respect to ϵ_o (see eq. (55b)) or where the series for f^o converges in equations (87)
d	dimensionless shock standoff distance, $\frac{r_s - r_b}{r_b}$
d^*	defined on page 42, determined by equation (199)
d^o	see equations (108), (110), and (118)
d_n	terms of expansion of d , equation (110)
F	magnified dependent variable, equation (60c)
F_n	functions of x, Y in equations (62) and (92b); in the blunt-body problem, functions of η ; see equations (152) and (159)
F_{21}, F_{22}	defined by equations (165) through (167)
f	generally a function of x, y, z, ϵ (see pages 5 and 17)
f^*	defined on pages 17, 18; also see page 34
f^i	inner function (eqs. (60b,c); also eqs. (88))

f^0 outer function (see pp. 21, 22, and 34)
 f_n generally functions of x, y in an asymptotic expansion (see p. 17)
 \tilde{f}^0 outer displacement dependent variable; see equations (68), (69), and (90b)
 \tilde{f}_n functions of \tilde{x}, \tilde{y} in the terms of an expansion, equations (69), (91); in the blunt body problem, see equations (123), (138)
 $G(\zeta)$ see equation (188)
 g arbitrary function; and function of ζ defined in equation (194) for the blunt body problem
 h specific enthalpy
 K_j constants defined on page 53
 k ρ_∞ / ρ_2
 L linear differential operator defined by equation (163)
 N Reynolds number defined in equation (189)
 P magnified inner pressure function defined on page 55
 P_n terms of the expansion (174)
 p pressure
 \bar{p} dimensionless pressure function (p. 38)
 \bar{p}^i inner pressure function defined on page 54
 \bar{p}^0 outer pressure function
 q^* defined on page 18
 q^i defined in equations (60b), (60c), (88); see also equations (132) and (134)
 q^0 defined on page 21; also equations (107) and (111)
 R Reynolds number, $\rho_2 V_\infty r_b / \mu_2$, equation (101)
 r radial distance measured from sphere center
 r_δ defined on page 42
 u, v dimensionless velocity components; see page 38

\vec{V}, V	flow velocity vector and magnitude of velocity, respectively
w	defined on page 22; see also pages 25 and 26
\tilde{x}	coordinate x , used along with \tilde{y} in the transformation to displacement variables (eqs. (67))
Y	magnified independent variable, equation (60a)
y	general independent variable; in the blunt body problem y is dimensionless and $r_b y$ is the distance from the body surface
y_δ	$e^\delta - 1$ (eq. (114)); $r_b y_\delta$ is the displacement distance from the body surface (see also p. 58)
\tilde{y}	independent displacement variable, $y - \delta$; see pages 27, 28, and 35
Z	magnified independent variable in the blunt body problem (eq. (131))
α	parameter in example problems 2 (p. 7) and 3 (p. 9); also used on page 31
β_n	see equations (157), (158), (161)
Δ	magnified displacement, δ/σ_δ
Δ_n	terms of expansion of Δ , equations (66) and (112)
Δ'	dimensionless magnified displacement of y coordinate, equation (114)
δ	general dimensionless displacement in boundary-layer type problems; see equations (22c), (24b), (34b), (59b), (64), and (112)
ϵ	generally a small parameter (see p. 5); in the blunt body problem, $\epsilon = R^{-1/2}$ (eq. (137))
ϵ_c	see page 25
ϵ_i, ϵ_o	see page 21
ζ	independent variable in blunt body problem (eqs. (100))
$\tilde{\zeta}$	displacement variable, $\zeta - \delta$
$\bar{\zeta}$	defined in equation (190)
η	independent variable for boundary-layer solution; see equation (153b)
λ	see equations (153c), (156)

μ	parameter in example problem 3 (p. 8); also viscosity coefficient in blunt body problem
ν	artificially inserted parameter, first appearing in condition (105c); discussed on pages 40, 41
ξ	independent variable for the boundary-layer solution; see equations (153a) and (156)
ρ	mass density
σ_1	order of magnitude of width of nonuniform region (see eq. (60a) and pp. 46, 47)
σ_2	order of magnitude of dependent variable in region of nonuniformity (see eq. (60c) and pp. 46, 47)
σ_δ	order of magnitude of δ ; usually $= \sigma_1$ (see eq. (65))
$\Phi(\varphi)$	function of integration defined on page 50
φ	angle measured from stagnation streamline, centered at sphere center (fig. 3)
$\tilde{\varphi}$	variable φ , used along with $\tilde{\xi}$ in transformation to displacement variables (eq. (119a))
Ψ	magnified inner stream function in the blunt body problem, equation (131)
Ψ_n	terms of expansion (133)
ψ	dimensionless Stokes stream function defined by equations (99)
ψ_n	terms of expansion (109)
ψ^i	defined on page 46
ψ^o	defined on page 41
$\tilde{\psi}^o$	defined in equation (119b)
$\tilde{\psi}_n$	terms of expansion of $\tilde{\psi}^o$, equation (119c)
ω	dimensionless vorticity, see page 45

Subscripts

z	value behind shock on stagnation streamline
∞	value in free stream

b	value at body surface or boundary
n	index denoting term of an expansion
s	value behind shock
φ, ζ	denote partial differentiation with respect to φ or ζ , respectively
φ, Z	denote partial differentiation with respect to φ or Z , respectively

Superscripts

i	inner function or function transformed from outer displacement variables to inner magnified variables
o	outer function or function transformed to outer displacement variables

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